Matrix Inversion Lemma

The Matrix Inversion Lemma is the equation

$$\left(A-B\cdot D^{-1}\cdot C\right)^{-1}=A^{-1}+A^{-1}\cdot B\cdot\left(D-C\cdot A^{-1}\cdot B\right)^{-1}\cdot C\cdot A^{-1} \tag{1}$$

Proof: We construct an augmented matrix *A*, *B*, *C*, and *D* and its inverse:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}=\begin{bmatrix} E & F \\ G & H \end{bmatrix} \tag{2}$$

We then construct the two products

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\cdot\begin{bmatrix} E & F \\ G & H \end{bmatrix}=\begin{bmatrix} A\cdot E+B\cdot G & A\cdot F+B\cdot H \\ C\cdot E+D\cdot G & C\cdot F+D\cdot H \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{3}$$

and

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix}\cdot\begin{bmatrix} A & B \\ C & D \end{bmatrix}=\begin{bmatrix} E\cdot A+F\cdot C & E\cdot B+F\cdot D \\ G\cdot A+H\cdot C & G\cdot B+H\cdot D \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{4}$$

Submatrices in (3) and (4) are broken out to form eight matrix equations:

$$A\cdot E+B\cdot G=I \tag{5}$$

$$A\cdot F+B\cdot H=0 \tag{6}$$

$$C\cdot E+D\cdot G=0 \tag{7}$$

$$C\cdot F+D\cdot H=I \tag{8}$$

$$E\cdot A+F\cdot C=I \tag{9}$$

$$E\cdot B+F\cdot D=0 \tag{10}$$

$$G\cdot A+H\cdot C=0 \tag{11}$$

$$G\cdot B+H\cdot D=I \tag{12}$$

Combining (5) through (12) in various orders gives two sets of equations for *E*, *F*, *G* and *H* from *A*, *B*, *C*, and *D*:

$$E=\left(A-B\cdot D^{-1}\cdot C\right)^{-1} \tag{13}$$

$$F=-\left(A-B\cdot D^{-1}\cdot C\right)^{-1}\cdot B\cdot D^{-1} \tag{14}$$

$$G=-D^{-1}\cdot C\cdot\left(A-B\cdot D^{-1}\cdot C\right)^{-1} \tag{15}$$

$$H=D^{-1}+D^{-1}\cdot C\cdot\left(A-B\cdot D^{-1}\cdot C\right)^{-1}\cdot B\cdot D^{-1} \tag{16}$$

and

$$E=A^{-1}+A^{-1}\cdot B\cdot\left(D-C\cdot A^{-1}\cdot B\right)^{-1}\cdot C\cdot A^{-1} \tag{17}$$

$$F=-A^{-1}\cdot B\cdot\left(D-C\cdot A^{-1}\cdot B\right)^{-1} \tag{18}$$

$$G=-\left(D-C\cdot A^{-1}\cdot B\right)^{-1}\cdot C\cdot A^{-1} \tag{19}$$

$$H=\left(D-C\cdot A^{-1}\cdot B\right)^{-1} \tag{20}$$

The proof is completed by combining either (13) and (17) or (16) and (20). Conditions are that all the involved inverses exist.