# A Brief Description of of Z Transforms

James K Beard

# Z Transforms and Digital Filters

- Difference equations
  - Z transforms classically used in analysis
  - Digital filter algebraic definitions
- Digital trackers
  - Stationary filters easily analyzed
  - Adaptive filters characterized and bounded
    - Bounds of parameters bound Z transform results
    - Z transform characterizations useful in interpretation

# Z Transform Definition

- Laplace transform analogy for uniformly sampled filters

$$F(s)=\int_0^\infty e(t)\cdot\exp(-s\cdot t)\cdot dt \qquad F(z)=\sum_{k=0}^{\infty} e(k)\cdot z^{-k}$$

- Function e(t) is series of impulses at $t_k=k\cdot T$
- Inversion integral

$$f(n)=\frac{1}{2\pi j}\cdot\oint F(z)\cdot z^{n-1}\cdot dz$$

- Variable change linking the analogy is

$$z=\exp(s\cdot T)$$

Where 1/T is sample rate

# Z Transform Pairs

| $f(n)$ | $F(z)$ |
|---|---|
| $a^n$ | $\dfrac{z}{z-a}$ |
| $f(n+a)$ | $z^a \cdot f(z) - \sum_{i=0}^{a} z^{a-i} \cdot f(i)$ |
| $f(n-a) \cdot u(n-a)$ | $z^{-a} \cdot F(z)$ |
| $\sum_{k=0}^{\infty} f_2(k) \cdot f_1(n-k)$ | $F_1(z) \cdot F_2(z)$ |

# Laplace-Stieltjes Transforms

- Form is

$$\int_0^\infty \exp\left(-s \cdot t\right) \cdot d\Phi\left(t\right)$$

- Where $\phi(t)$
  - Constant except for steps at $t_n = nT$
  - Size of steps is f(n)
- A Z transform for mathematicians

# Digital Filters and Difference Equations

- Simple digital filter

$$y(n+1) = (1-a) \cdot x(n) + a \cdot y(n)$$

- Taking Z transform of both sides:

$$z \cdot Y(z) - z \cdot y(0) = (1-a) \cdot X(z) + a \cdot Y(z)$$

- Solving for y(z)

$$Y(z) = \frac{z}{z-a} \cdot y(0) + \frac{1-a}{z-a} \cdot X(z)$$

# Transfer Function

- First term
  - Ringing or settling from initial conditions
  - Decays to zero in stable filter ($|a|<1$)
  - Ignored in transfer function from input
- Transfer function

$$G(z)=\frac{Y(z)}{X(z)}=\frac{1-a}{z-a}$$

# Initial and Final Values of f(n)

- From definition

$$F(z)=\sum_{n=0}^{\infty} z^{-n}\cdot f(n)$$

- Initial value is easy

$$f(0)=\lim_{z\to\infty} F(z)$$

- Final value from

$$Z\Big(f(n+1)-f(n)\Big)=z\cdot F(z)-z\cdot f(0)-F(z)$$

# Final Value (Continued)

- The logic is

$$\left(z-1\right)\cdot F\left(z\right)-f\left(0\right)=$$

$$\lim_{p\to\infty}\sum_{n=0}^{p}\left(f\left(n+1\right)-f\left(n\right)\right)\cdot z^{-n}$$

- Taking a second limit $z\to 1$

$$\lim_{n\to\infty}f\left(n\right)=\lim_{z\to 1}\left(z-1\right)\cdot F\left(z\right)$$

# Noise Variance Propagation

- Variance difference equation

$$\sigma_y^2(i+1) = (1-a)^2 \cdot \sigma_x^2(i) + a^2 \cdot \sigma_y^2(i)$$

- Z transform of output variance

$$V_y(z) = \frac{z}{z-a^2} \cdot \sigma_y^2(0) + \frac{(1-a)^2}{z-a^2} \cdot V_x(z)$$

- Steady state output variance (why?)

$$\sigma_y^2 = \frac{1-a}{1+a} \cdot \sigma_x^2$$

# Frequency Response

- Follows from variable changes

$$z = \exp(s \cdot T), s = j \cdot \omega$$

- Z transform is then

$$F(\omega) = \sum_{n=0}^{\infty} f(n) \cdot \exp(-j \cdot \omega \cdot T)$$

- This is a frequency response
  - Same equation as Fourier summation
  - From Z transform for z on unit circle (|z|=1)

# Generalization Using Vectors

- Order N digital filter difference equation

$$\underline{y}(n+1) = A \cdot \underline{y}(n) + B \cdot \underline{x}(n)$$

- And the Z transform is

$$z \cdot \underline{Y}(z) - z \cdot \underline{y}(0) = A \cdot \underline{Y}(z) + B \cdot \underline{X}(z)$$

- Solving for y(z)

$$\underline{Y}(z) = -\left[A - z \cdot I\right]^{-1} \cdot \left[B \cdot \underline{X}(z) + z \cdot \underline{y}(0)\right]$$

# The Poles

- Denominator polynomial
  - Determinant $|A - z \cdot I|$
  - Order N
- Poles of $\underline{Y}(z)$ are
  - Roots of denominator polynomial
  - The characteristic values of matrix A
  - Filter stable is all magnitudes less than 1

# Example

- Constant velocity model

$$\underline{x}=\begin{bmatrix} Position \\ Velocity \end{bmatrix},\ A=\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

- Z transform of output is

$$\underline{Y}(z)=\begin{bmatrix} \frac{1}{z-1} & \frac{T}{(z-1)^2} \\ 0 & \frac{1}{z-1} \end{bmatrix}\cdot\left[B\cdot\underline{X}(z)+z\cdot\underline{y}(0)\right]$$

# Decoupling the Matrix Equation

- Make a variable change

$$\underline{w}(n) = C \cdot \underline{y}(n)$$

- The new difference equation is

$$\underline{w}(n+1) = C \cdot A \cdot C^{-1} \cdot \underline{w}(n) + C \cdot B \cdot \underline{x}(n)$$

- Select C as the characteristic vector matrix of A

$$C \cdot A \cdot C^{-1} = \Lambda$$

# The Form of C and $\Lambda$

- C is characteristic vector matrix
  - Columns of $C^{-1}$ are characteristic vectors
  - If A is symmetric, then $C^{-1} = C^T$
- The matrix $\Lambda$ is diagonal if
  - No characteristic values are repeated
- Repeated characteristic values
  - Matrix $\Lambda$ is not diagonal
  - Submatrices of Jordan canonical form

# Characteristic Values

- Any characteristic value λ and vector <u>c</u>

$$A \cdot \underline{c} = \lambda \cdot \underline{c}$$

- Jordan canonical form submatrices

$$\Lambda_R = \begin{bmatrix} \lambda_R & 1 & 0 \\ 0 & \lambda_R & 1 \\ 0 & 0 & \lambda_R \end{bmatrix}$$

# Time Functions for Multiple Characteristic Values

- Terms appearing in the Z transform

$$\frac{z^i}{(z-a)^i}, 1 \le i \le N_R$$

- Corresponding time series terms

$$n^{i-1} \cdot a^n$$

# Form of Λ Matrix

- Form of inverse of [z·I – Λ]

$$\left[z \cdot I - \Lambda_R\right]^{-1} = \begin{bmatrix} \dfrac{1}{z-\lambda_R} & \dfrac{1}{\left(z-\lambda_R\right)^2} & \dfrac{1}{\left(z-\lambda_R\right)^3} \\ 0 & \dfrac{1}{z-\lambda_R} & \dfrac{1}{\left(z-\lambda_R\right)^2} \\ 0 & 0 & \dfrac{1}{z-\lambda_R} \end{bmatrix}$$

# Useful Books

- “Introduction to matrix analysis” (Second Edition), Richard Bellman, McGraw-Hill, 1970
- “Operational mathematics,” R. V. Churchill, McGraw-Hill (1958)
- “Theory and application of the Z transform method,” E. I. Jury, John Wiley & Sons, Inc. (1964)