

A Brief Description of of Z Transforms

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Z Transforms and Digital Filters

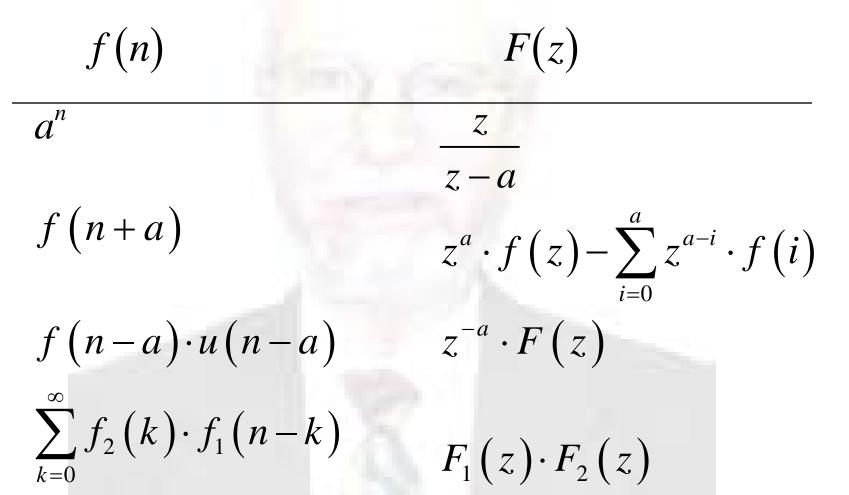
- Difference equations
 - Z transforms classically used in analysis
 - Digital filter algebraic definitions
- Digital trackers
 - Stationary filters easily analyzed
 - Adaptive filters characterized and bounded
 - Bounds of parameters bound Z transform results
 - Z transform characterizations useful in interpretation

Z Transform Definition

- Laplace transform analogy for uniformly sampled filters $F(s) = \int_{0}^{\infty} e(t) \cdot \exp(-s \cdot t) \cdot dt$ $F(z) = \sum_{k=0}^{\infty} e(k) \cdot z^{-k}$
- Function e(t) is series of impulses at t_k=k·T
- Inversion integral $f(n) = \frac{1}{2\pi j} \cdot \oint F(z) \cdot z^{n-1} \cdot dz$ • Variable change linking the apples
- Variable change linking the analogy is $z = \exp(s \cdot T)$

Where 1/T is sample rate

Z Transform Pairs



Laplace-Stieltjes Transforms

• Form is

$$\int_{0}^{\infty} \exp(-s \cdot t) \cdot d\Phi(t)$$

- Where $\phi(t)$
 - Constant except for steps at $t_n = nT$
 - Size of steps is f(n)
- A Z transform for mathematicians

Digital Filters and Difference Equations

- Simple digital filter $y(n+1) = (1-a) \cdot x(n) + a \cdot y(n)$
- Taking Z transform of both sides: $z \cdot Y(z) - z \cdot y(0) = (1 - a) \cdot X(z) + a \cdot Y(z)$
- Solving for y(z)

$$Y(z) = \frac{z}{z-a} \cdot y(0) + \frac{1-a}{z-a} \cdot X(z)$$

Transfer Function

- First term
 - Ringing or settling from initial conditions
 - Decays to zero in stable filter (|a| < 1)
 - Ignored in transfer function from input
- Transfer function

$$G(z) = \frac{Y(z)}{X(z)} = \frac{1-a}{z-a}$$

Initial and Final Values of f(n)

From definition

 $F(z) = \sum_{n=0}^{\infty} z^{-n} \cdot f(n)$

Initial value is easy

 $f(0) = \lim_{z \to \infty} F(z)$

Final value from

$$Z(f(n+1)-f(n)) = z \cdot F(z) - z \cdot f(0) - F(z)$$

Final Value (Continued)

• The logic is

$$(z-1) \cdot F(z) - f(0) =$$

$$\lim_{p \to \infty} \sum_{n=0}^{p} (f(n+1) - f(n)) \cdot z^{-n}$$

• Taking a second limit $z \rightarrow 1$

$$\lim_{n\to\infty} f(n) = \lim_{z\to 1} (z-1) \cdot F(z)$$

Noise Variance Propagation

- Variance difference equation $\sigma_y^2(i+1) = (1-a)^2 \cdot \sigma_x^2(i) + a^2 \cdot \sigma_y^2(i)$
- Z transform of output variance $V_{y}(z) = \frac{z}{z-a^{2}} \cdot \sigma_{y}^{2}(0) + \frac{(1-a)^{2}}{z-a^{2}} \cdot V_{x}(z)$
- Steady state output variance (why?) $\sigma_y^2 = \frac{1-a}{1+a} \cdot \sigma_x^2$

Frequency Response

Follows from variable changes

 $z = \exp(s \cdot T), s = j \cdot \omega$

Z transform is then

$$F(\omega) = \sum_{n=0}^{\infty} f(n) \cdot \exp(-j \cdot \omega \cdot T)$$

- This is a frequency response
 - Same equation as Fourier summation
 - From Z transform for z on unit circle (|z|=1)

Generalization Using Vectors

- Order N digital filter difference equation $\underline{y}(n+1) = A \cdot \underline{y}(n) + B \cdot \underline{x}(n)$
- And the Z transform is $z \cdot \underline{Y}(z) - z \cdot \underline{y}(0) = A \cdot \underline{Y}(z) + B \cdot \underline{X}(z)$
- Solving for y(z)

 $\underline{Y}(z) = -\left[A - z \cdot I\right]^{-1} \cdot \left[B \cdot \underline{X}(z) + z \cdot \underline{y}(0)\right]$

The Poles

- Denominator polynomial
 - Determinant |A z·I|
 - Order N
- Poles of $\underline{Y}(z)$ are
 - Roots of denominator polynomial
 - The characteristic values of matrix A
 - Filter stable is all magnitudes less than 1

Example

Constant velocity model

$$\underline{x} = \begin{bmatrix} Position \\ Velocity \end{bmatrix}, A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

Z transform of output is

$$\underline{Y}(z) = \begin{bmatrix} \frac{1}{z-1} & \frac{T}{(z-1)^2} \\ 0 & \frac{1}{z-1} \end{bmatrix} \cdot \begin{bmatrix} B \cdot \underline{X}(z) + z \cdot \underline{y}(0) \end{bmatrix}$$

Decoupling the Matrix Equation

- Make a variable change $\underline{w}(n) = C \cdot \underline{y}(n)$
- The new difference equation is $\underline{w}(n+1) = C \cdot A \cdot C^{-1} \cdot \underline{w}(n) + C \cdot B \cdot \underline{x}(n)$
- Select C as the characteristic vector matrix of A

$$C \cdot A \cdot C^{-1} = \Lambda$$

The Form of C and Λ

- C is characteristic vector matrix
 - Columns of C⁻¹ are characteristic vectors
 - If A is symmetric, then $C^{-1} = C^{T}$
- The matrix Λ is diagonal if
 - No characteristic values are repeated
- Repeated characteristic values
 - Matrix Λ is not diagonal
 - Submatrices of Jordan canonical form

Characteristic Values

- Any characteristic value λ and vector \underline{c} $A \cdot \underline{c} = \lambda \cdot \underline{c}$
- Jordan canonical form submatrices

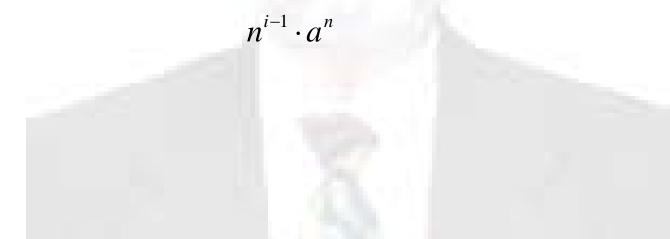
$$\Lambda_{R} = \begin{bmatrix} \lambda_{R} & 1 & 0 \\ 0 & \lambda_{R} & 1 \\ 0 & 0 & \lambda_{R} \end{bmatrix}$$

Time Functions for Multiple Characteristic Values

• Terms appearing in the Z transform

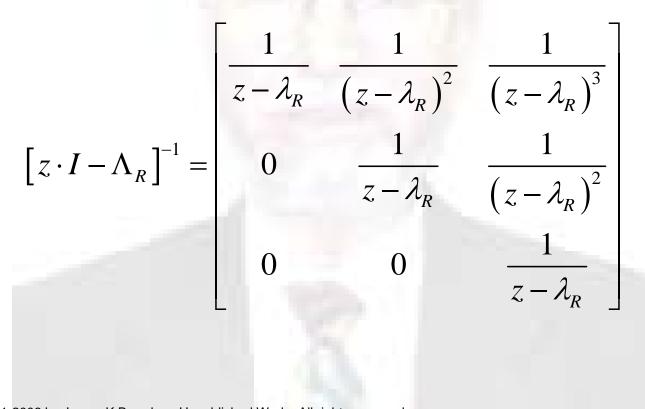
$$\frac{z^i}{\left(z-a\right)^i}, 1 \le i \le N_R$$

Corresponding time series terms



Form of Λ Matrix

• Form of inverse of $[z \cdot I - \Lambda]$



Useful Books

- "Introduction to matrix analysis" (Second Edition), Richard Bellman, McGraw-Hill, 1970
- "Operational mathematics," R. V. Churchill, McGraw-Hill (1958)
- "Theory and application of the Z transform method," E. I. Jury, John Wiley & Sons, Inc. (1964)