# Costas Arrays

What, Why, How, and When

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# Tonight's Topics

- Definition of Costas arrays
- Significance of Costas arrays
- Methods to obtain Costas arrays
- Principal uses of Costas arrays
  - Waveform example
  - Other
- The future of Costas arrays
- Conclusions, References

# Definition of Costas Arrays

- Costas arrays are permutation matrices with an added constraint, the Costas condition
- Costas condition: When a Costas array matrix and a replica of itself are overlaid, the replica with an offset of an integral number of rows and columns, only one “1” overlays another “1”
- This models a signal with a Doppler shift being processed with a matched filter

# Analyzing Costas Arrays

- Two common representations
  - As a *permutation matrix*
  - A *row vector*, value at each column position designates the positions of the “1”s
- The most often used tool is the *difference triangle*
  - First row: row vector of row indices
  - Other rows: Differences between row indices

# The Difference Triangle

- Row zero is the sequence of row indices
  - N is the order of the Costas array
  - Has N elements c(j)
- Row 1:
  - Column j is the difference c(j+1)-c(j)
  - Has N-1 elements
- Row i:
  - Column j is the difference c(j+i)-c(j)
  - Has N-i elements

# Example of Difference Triangle

| Costas array | 3 | 4 | 2 | 1 | 5 |
|---|---|---|---|---|---|
| Row 1 | 1 | -2 | -1 | 4 | |
| Row 2 | -1 | -3 | 3 | | |
| Row 3 | -2 | 1 | | | |
| Row 4 | 2 | | | | |



Array is a permutation if the first row has no duplicate entries and all are between 1 and N (or between 0 and N-1)

Shifting replica of Costas array right 1 row and down 1 column overlays ones at (3,1) and (4,2)

Similarly, each element of the difference matrix provides row and column shifts that Overlay ones

Thus, the Costas condition is equivalent to requiring that values appear only once in any given row

# Difference Vectors

- A *difference vector* is the difference in (row,col) coordinates between two ones in the Costas array matrix
- Each element in the difference triangle corresponds to a difference vector
  - Row i, difference triangle entry in column j is d(i,j)
  - Difference vector is (i,-d(i,j))
  - Another difference vector is its negative, -(i,-d(i,j))
- Costas condition is equivalent to requiring that no two difference vectors be equal

# Discrete Ambiguity Function

- A Discrete Ambiguity Function (DAF) is the number of overlaying ones as a function of rows and columns shifted
- Simple construction of DAF:
  - Size is (2N-1) by (2N-1)
  - Center is the order, N
  - From difference triangle; ones at (i,-d(i,j)) and at -(i,-d(i,j))
  - Other squares are zero
- Difference vectors in CA are positions of ones in the DAF

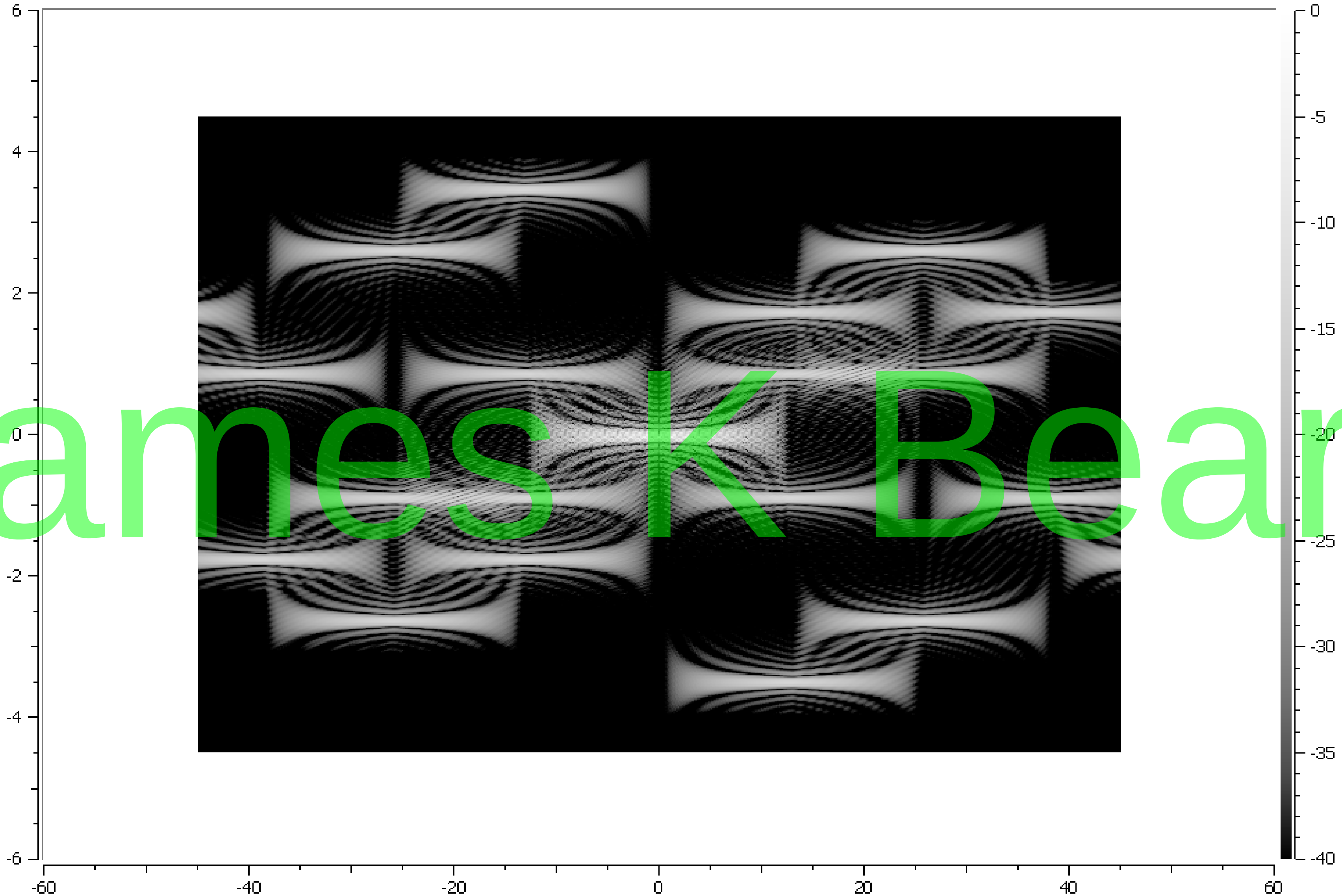
# Significance of DAF

- Ambiguity function of waveform
  - Coherent sum of ambiguity functions of single pulses
  - Relative positions and amplitudes of each such ambiguity function are given by positions and numbers in the DAF
- Watch out for a sign convention
  - DAF row indices increase as position moves up
  - Most matrices such as Costas arrays are represented with row indices that increase as position moves down
  - Mismatch will result in an upside-down DAF

# Example of DAF

| | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| 4 | | | | 1 | | | | | |
| 3 | | | 1 | | | | 1 | | |
| 2 | 1 | | | | | 1 | | 1 | |
| 1 | | 1 | | 1 | | 1 | 1 | | |
| 0 | | | | | 5 | | | | |
| -1 | | | 1 | 1 | | 1 | | 1 | |
| -2 | | 1 | | 1 | | | | | 1 |
| -3 | | | 1 | | | | 1 | | |
| -4 | | | | | | 1 | | | |

# Ambiguity Function using CA



# Significance of Costas Arrays

- It was always about waveforms from the beginning
  - First Costas array definition by John Costas for Project MEDIOR was a frequency shift scheme for sonar waveforms
  - Desired effect is that no combination of range and Doppler offset results in more than one overlaid pulse
- Huge differences in hydroacoustic and electromagentic waves requires fundamental differences in processing

# Huge EM and SONAR Differences

- Velocity of propagation: 3E8 m/s vs. 1500 m/s
- Typical frequencies: 10 GHz vs. 40 kHz
- Coherency versus medium
  - Sea water: poor over widely separated frequencies
  - RF: generally excellent except in some cases in the ionosphere
- Acoustic dispersion is high in sea water

# Costas Arrays in Waveforms

- Bandwidth spreading schemes in
  - Radar
  - Sonar
  - Communications
- Pseudorandom sequences for use in digital coding
- Other:  Add nearly invisible spots at the black level in digital photographs as a digital watermarking technique

# Modern Radar Waveforms

- We will give elementary examples
- Practical examples involve layered techniques
- Good reference for use of Costas arrays in layered methods for high performance radar waveforms:
  - *Radar Signals*, by Nadav Levanon and Eli Mozeson

# Fundamental Trade Parameters

- For an order of N and simple CW pulses
  - CW pulse length $\tau$
  - Time-bandwidth product is $N^2$
  - Peak to sidelobe ratio is $20\log_{10} N$
  - Shading over the N pulses is not helpful
- Width of each frequency channel is about $\frac{1}{\tau}$
- Ambiguity function of full waveform is coherent sum of ambiguity functions of the CW pulses

# Example

- Elementary example: Simple CW chips
- Two-way range resolution $\Delta R = \frac{1}{N} * \frac{c * \tau}{2}$
- Bandwidth $BW \approx \frac{N}{\tau}$
- Frequency Resolution $\Delta F \approx \frac{1}{N * \tau}$
- Ambiguity function sidelobes $20 * \log_{10} N$
  - Degraded by sidelobes of chip ambiguity functions
  - Accurate estimates determined by simulations

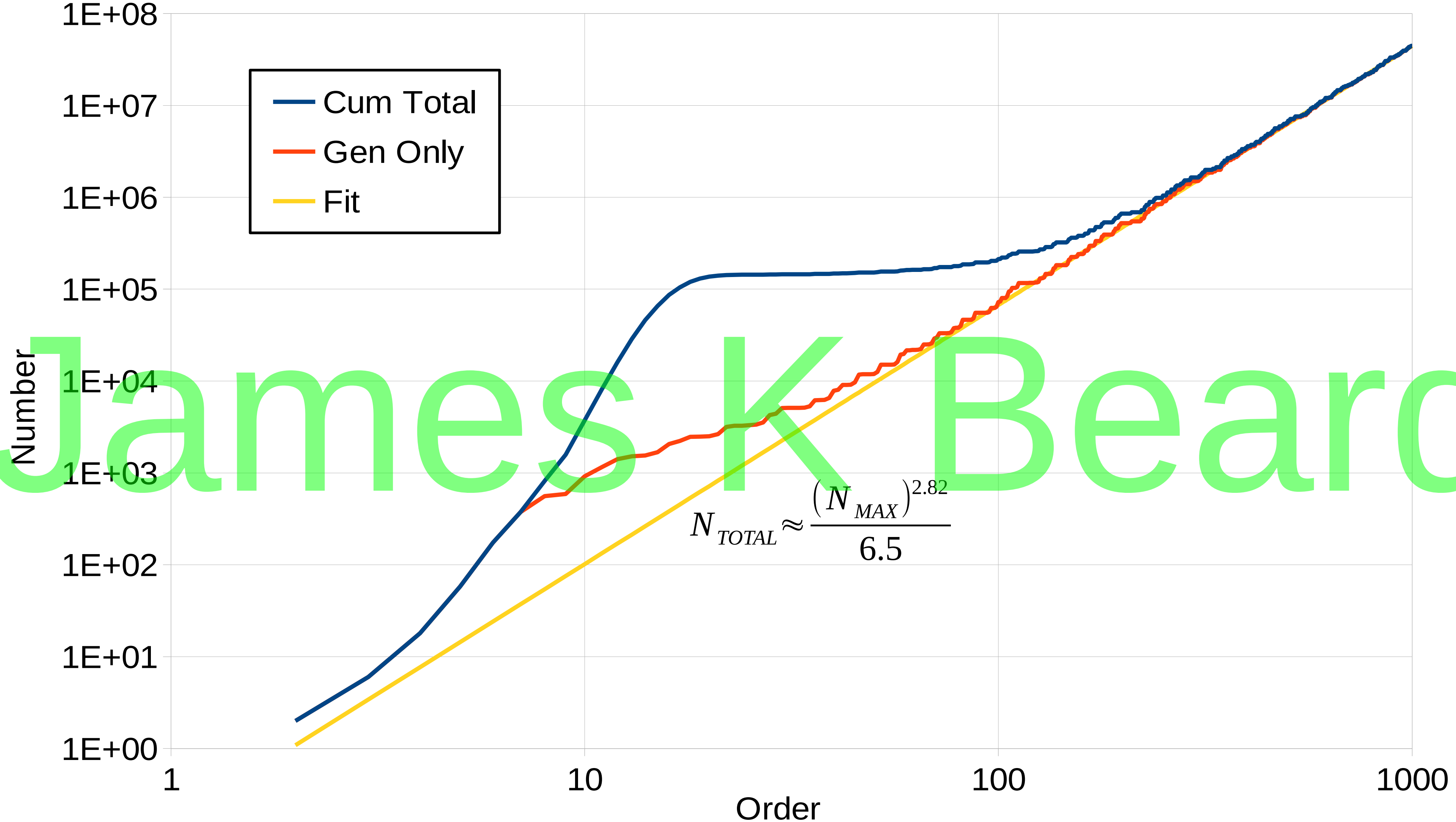
# Methods to Obtain Costas Arrays

- Comprehensive search
  - Simple and fast for orders up through about 20
  - Computation time increases by about a factor of five for each order increase of one
- Number-theoretic generators
  - Welch generator, $j = \alpha^{i-1} \left( mod\ p \right)$
  - Lempel-Golomb generator, $\alpha^i + \beta^j = 1 \in GF\left(p^k\right)$
  - Taylor generalizations; other generalizations

# Online Database

- IEEE DataPort https://ieee-dataport.org/
  - DOI 10.21227/H21P42
  - Creative Commons Attribution license
- All known Costas arrays to order 1030
- Separate searched database for orders to 29
- Windows GUI utility for search, extraction, analysis; Linux version coming soon

# Numbers of Costas Arrays



# Principal Uses of Costas Arrays

- Waveforms
  - Pseudodrandom frequency hopping scheme
  - Single or repeating waveform
  - Radar or communications
- Reasons for frequency hopping
  - Shared bandwidth – mutual platform interference in radars, communications, cell phones, …
  - Robustness against interference
  - Lower probability of detection of emissions

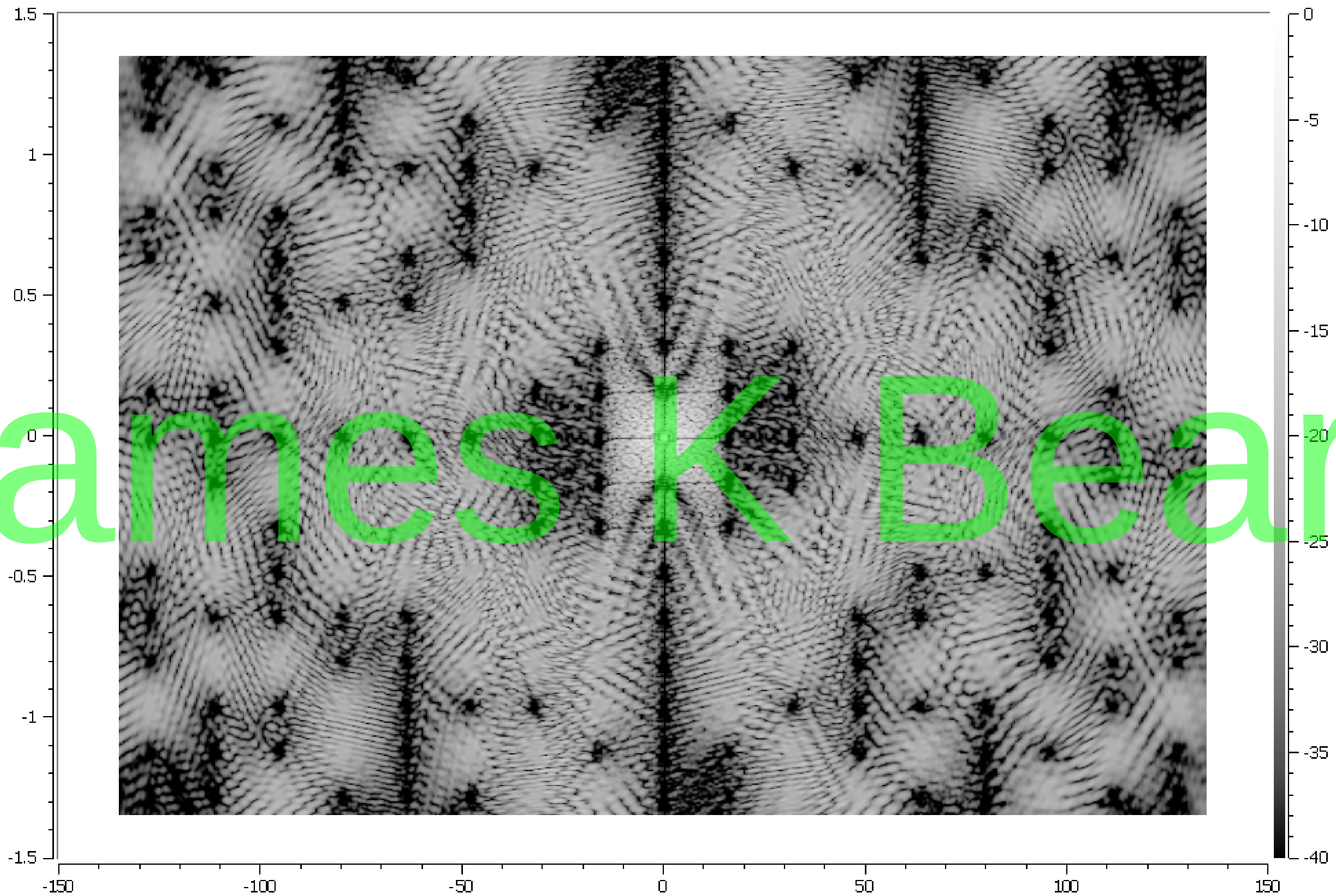
# Example Waveform

- Two variations
  - Order 14 Costas array, chips simple CW pulses
  - Same Costas array, chips are chirped
- One order 14 Costas array
  - The only Costas array with only two ones in the central 5X5 square
  - Symmetrical, transposition and rotation produces only four “siblings”
  - Order 14, TW product is 196, a good match for some radar applications

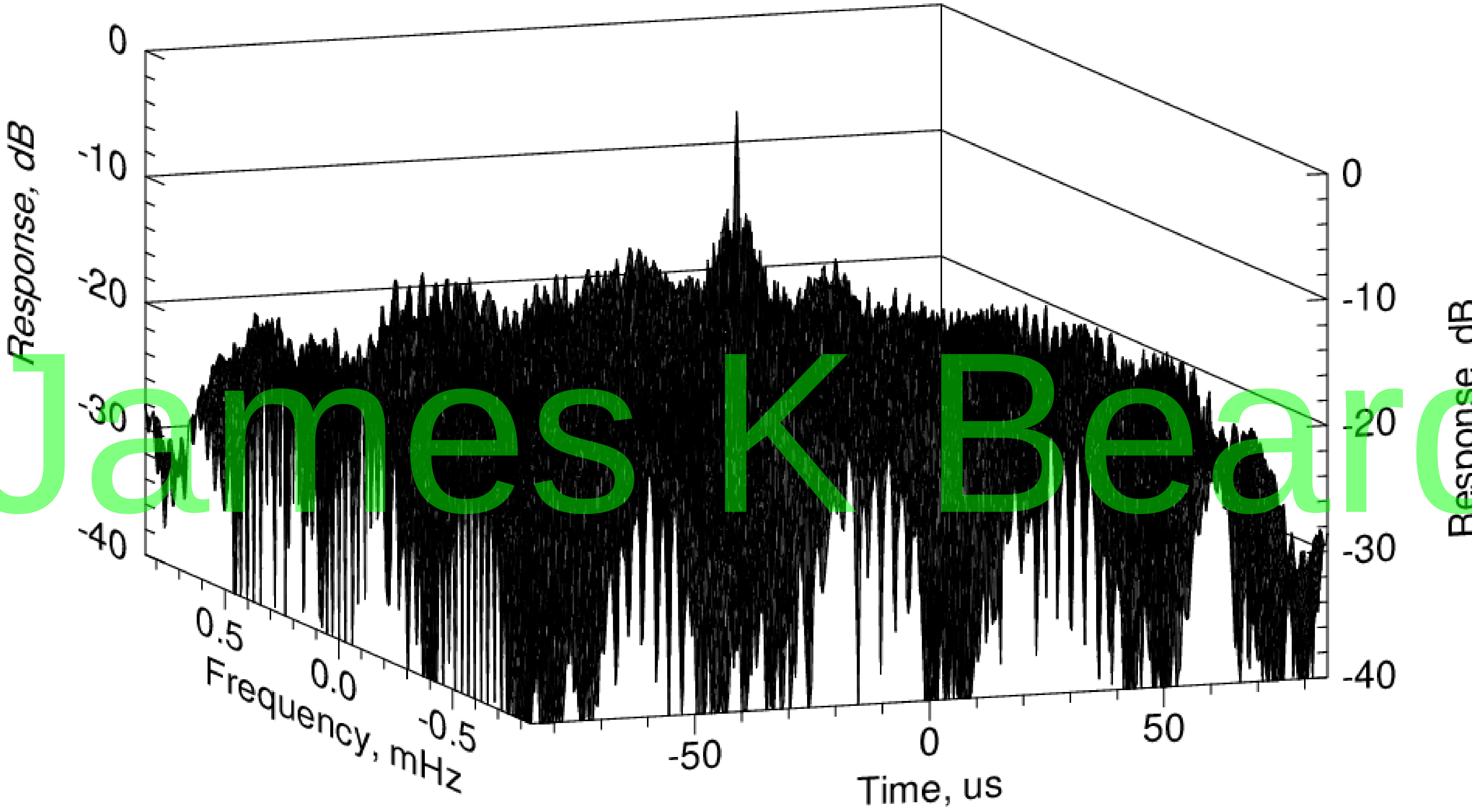
# Waveform Parameters

- Costas array order 14
  - 20 log10(N) is about 22.3 dB
  - Row indices are {8,13,3,6,10,2,14,5,11,7,1,12,9,4}
- Chip pulse length 10 μs
- Derivative parameters
  - Bandwidth 1.4 MHz
  - Sample rate 2.9 MHz complex
  - Model data length 1024 complex samples

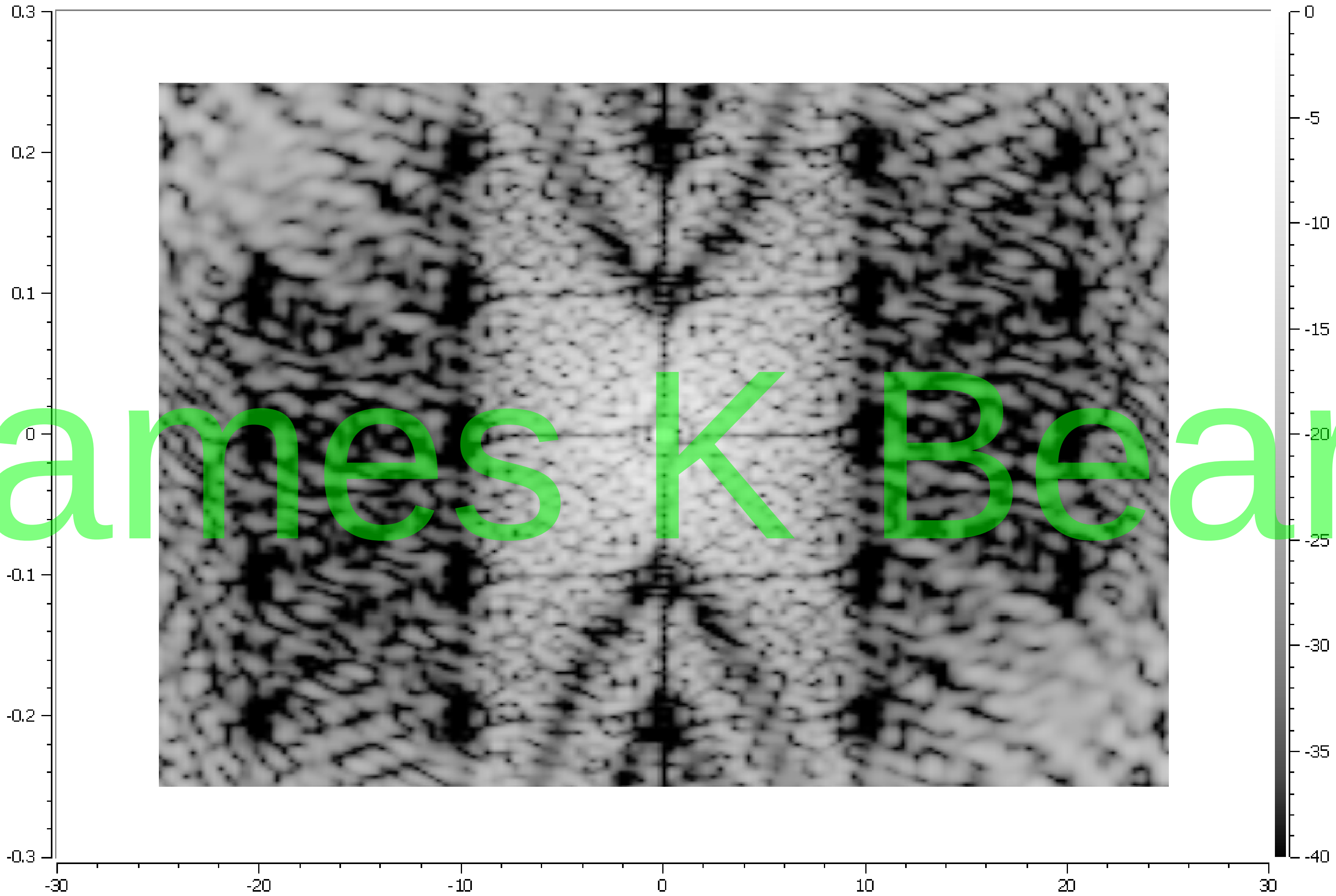
# Ambiguity Function Contour



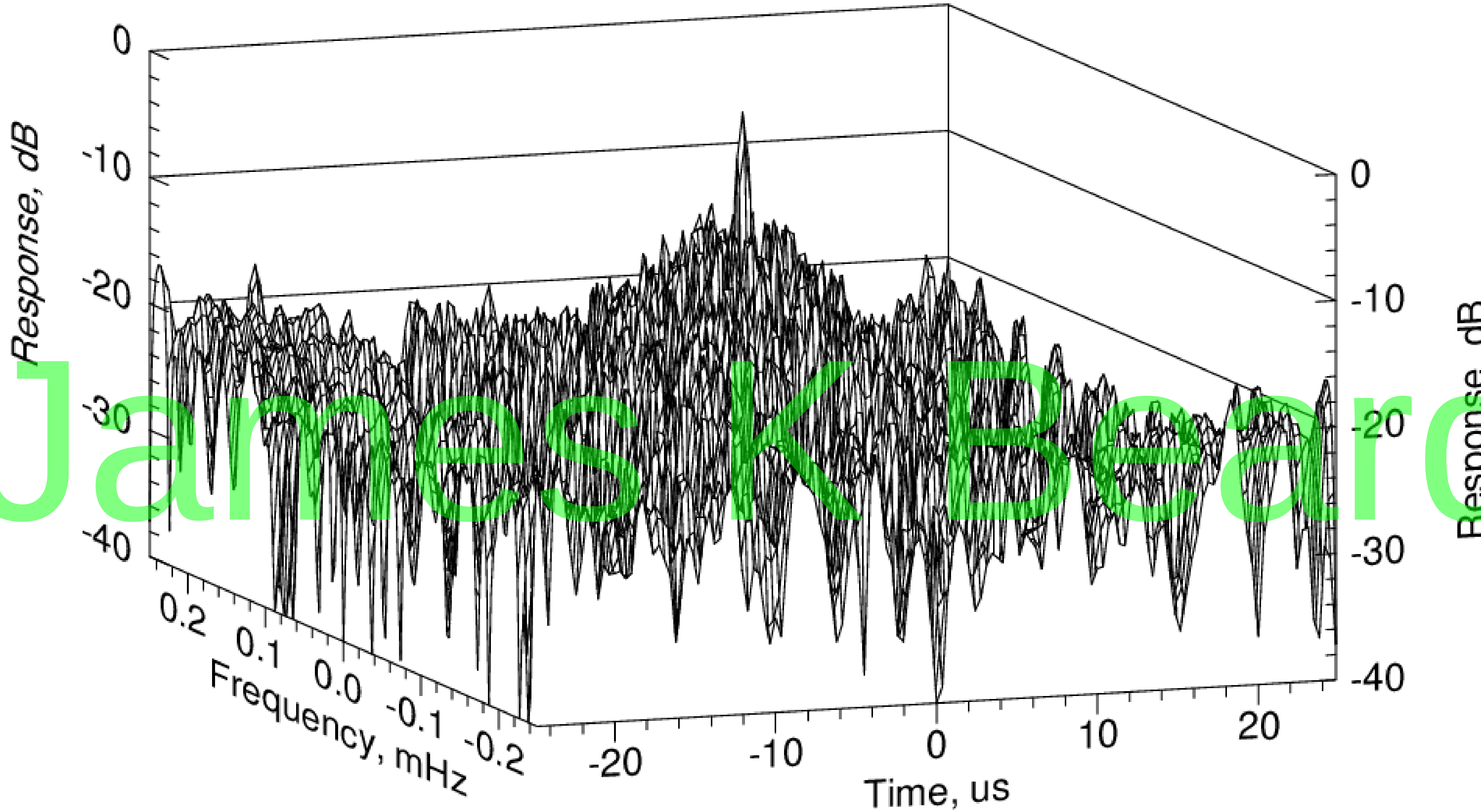
# Ambiguty Function Mesh Plot



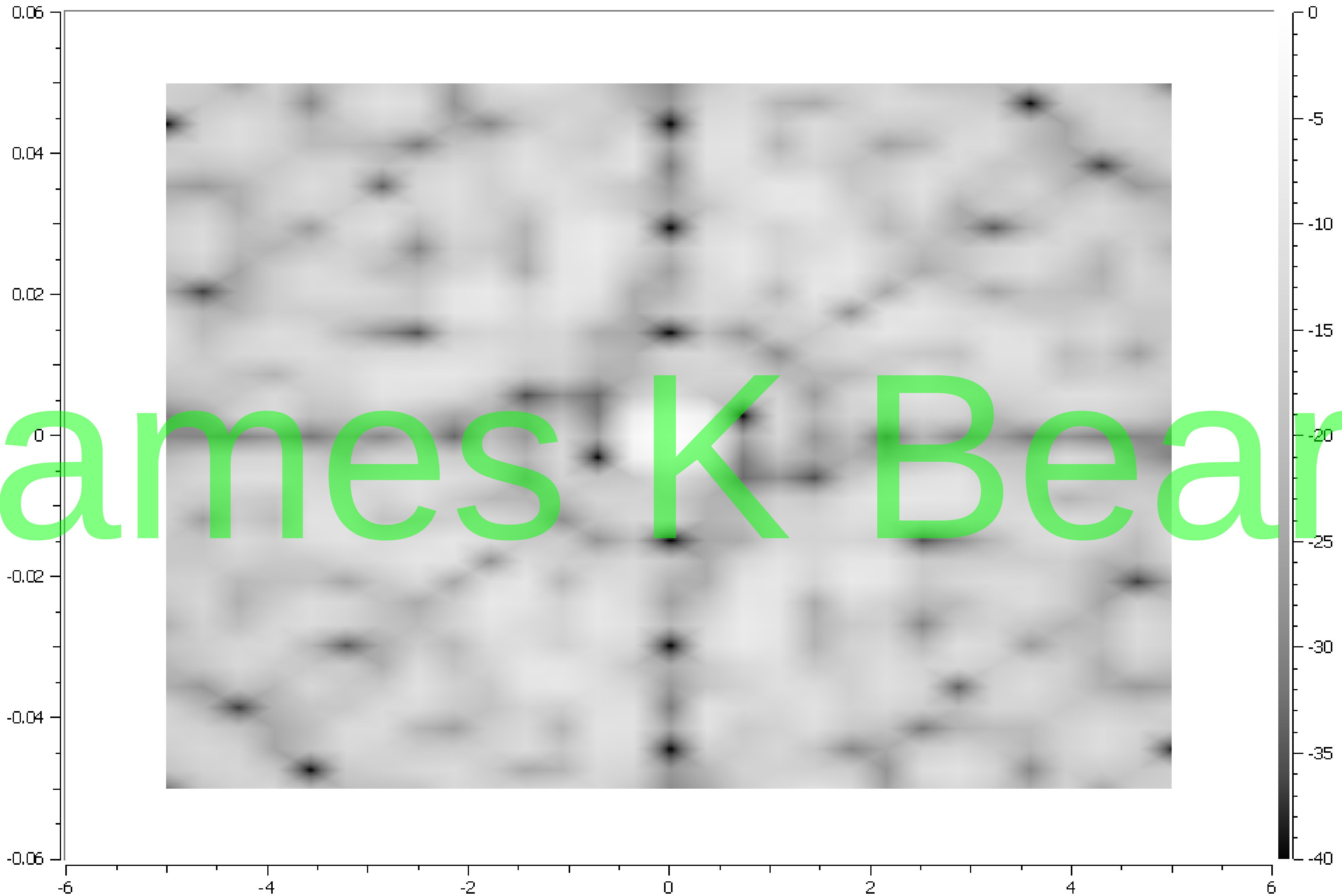
# Central 5X5 Contour



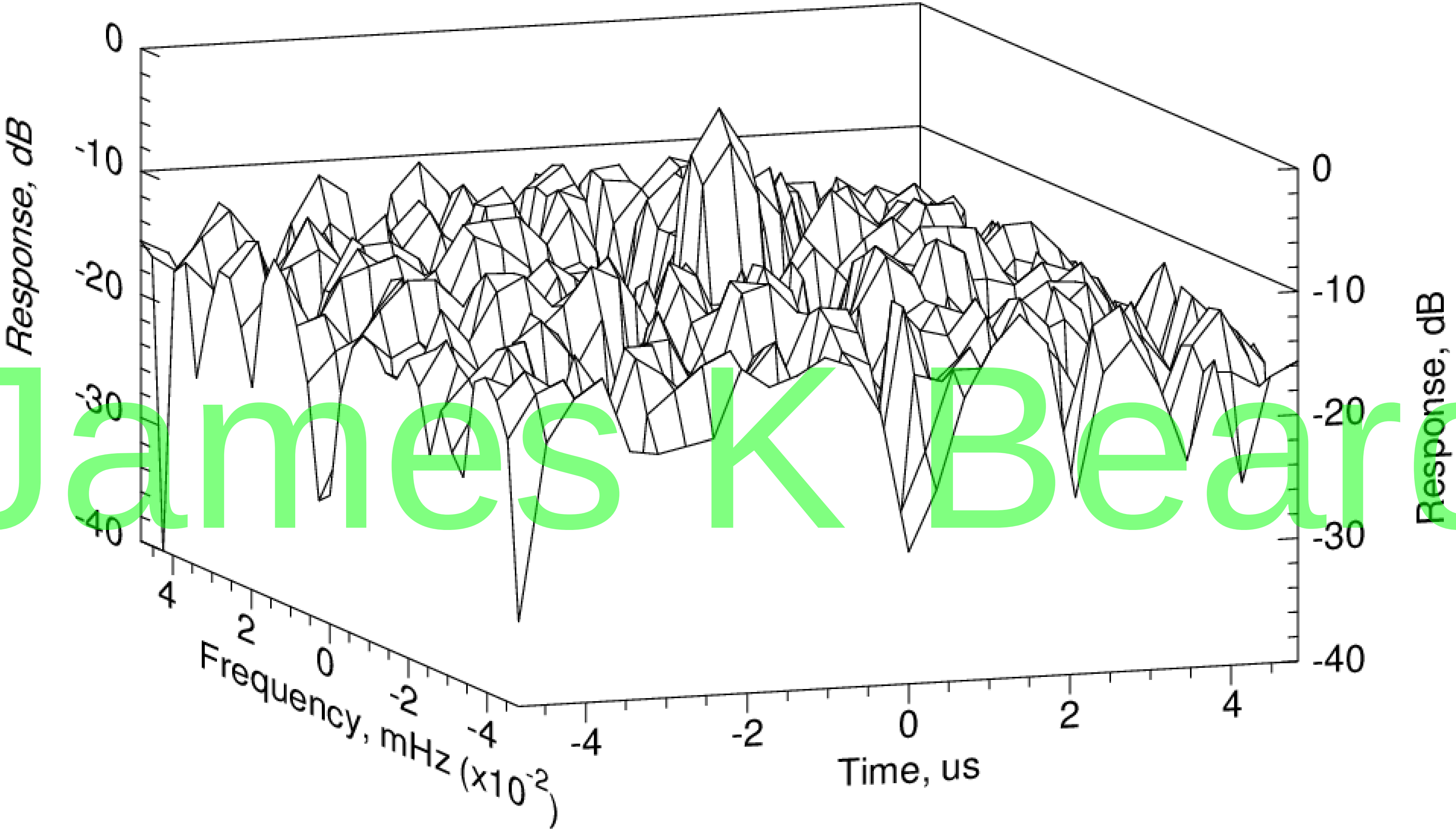
# Central 5X5 Mesh Plot



# Central Square Contour



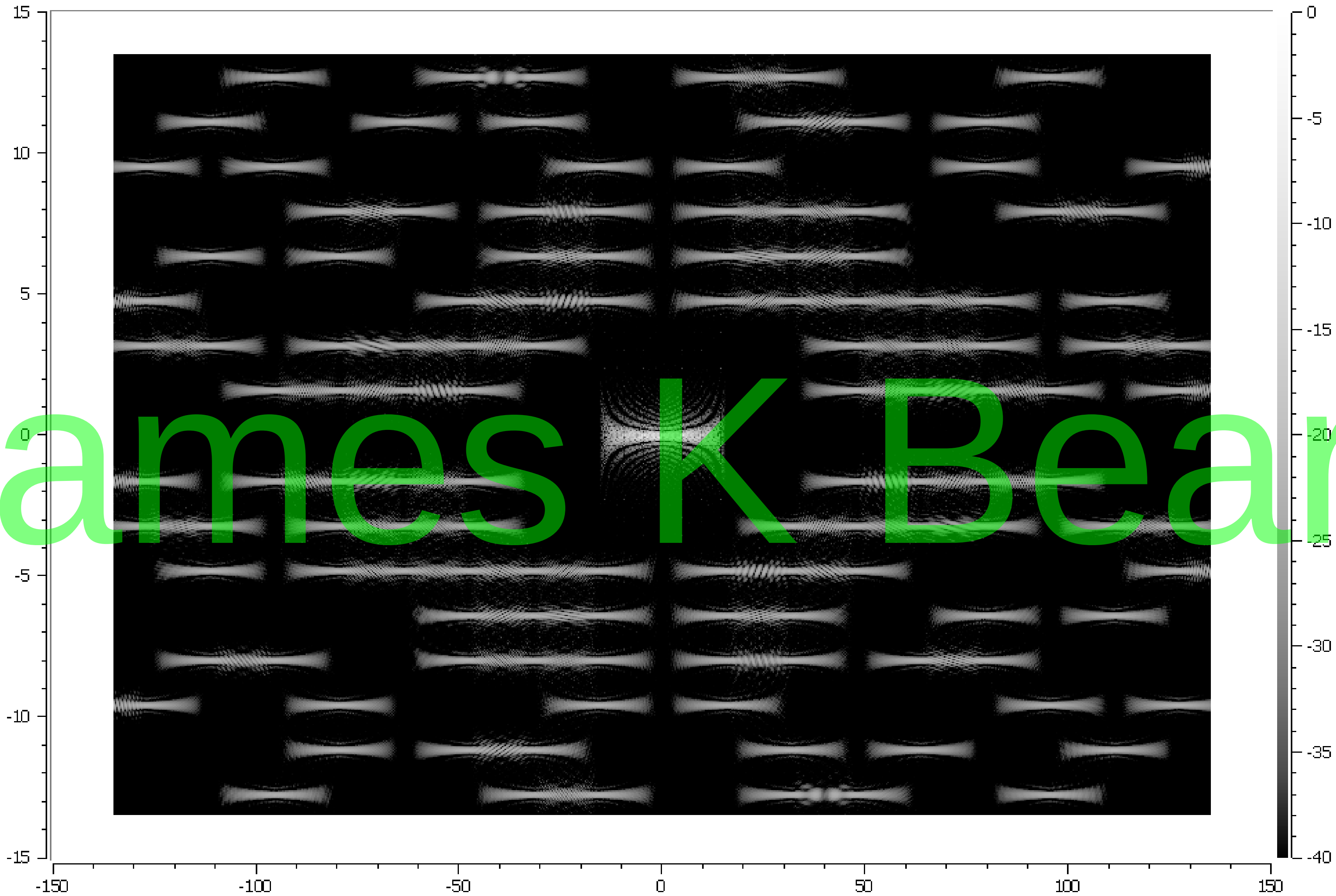
# Central Square Mesh Plot



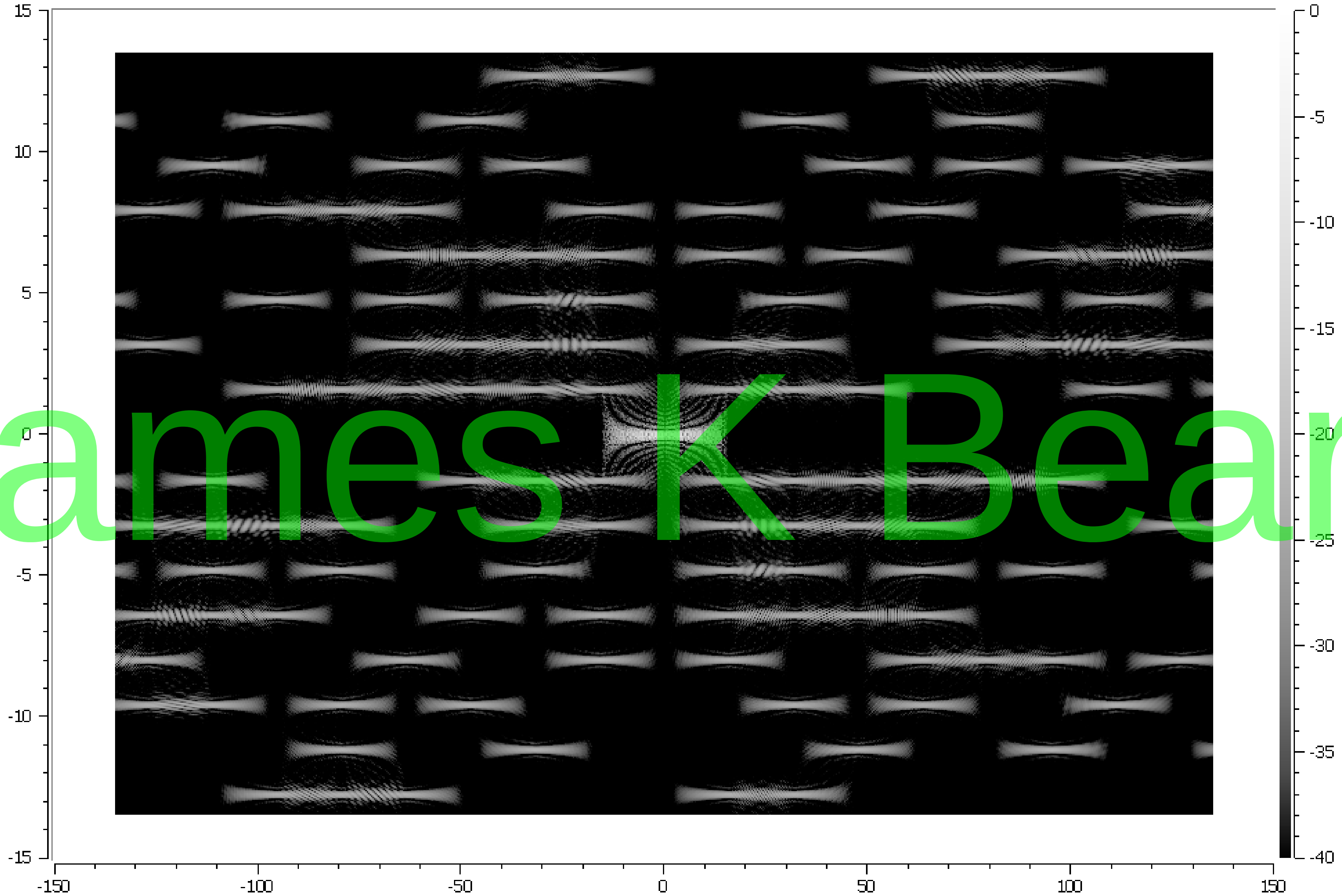
# Second Example Waveform

- Same Costas array
  - {8,13,3,6,10,2,14,5,11,7,1,12,9,4}
- Same chip pulse length, 10 μs
- Chip upchirp 1 MHz
- Derivative parameters
  - Bandwidth 14.1 MHz
  - Sample rate 28.2 MHz complex
  - Model data length 8192 complex samples

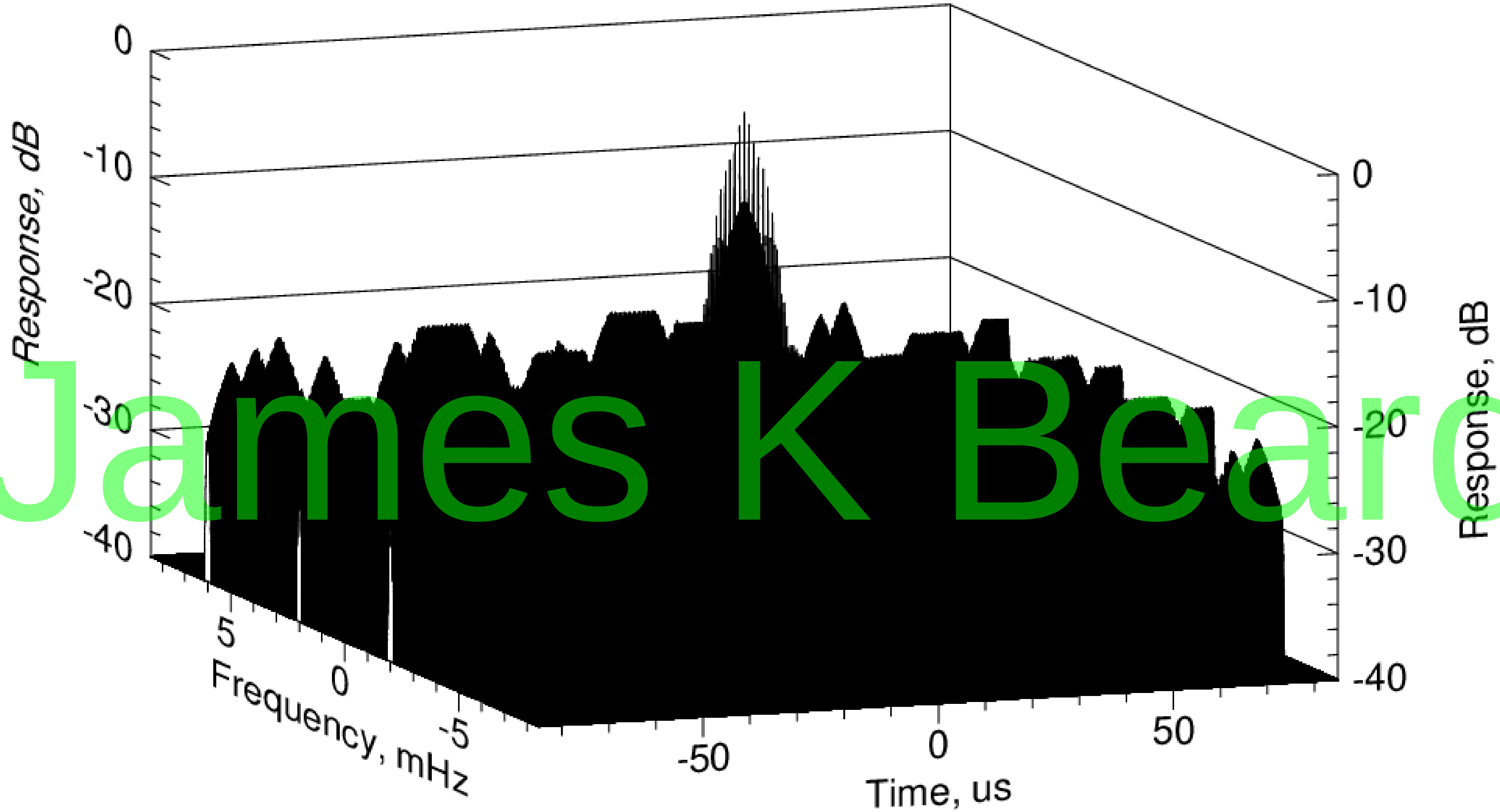
# Ambiguity Function w/Chirp



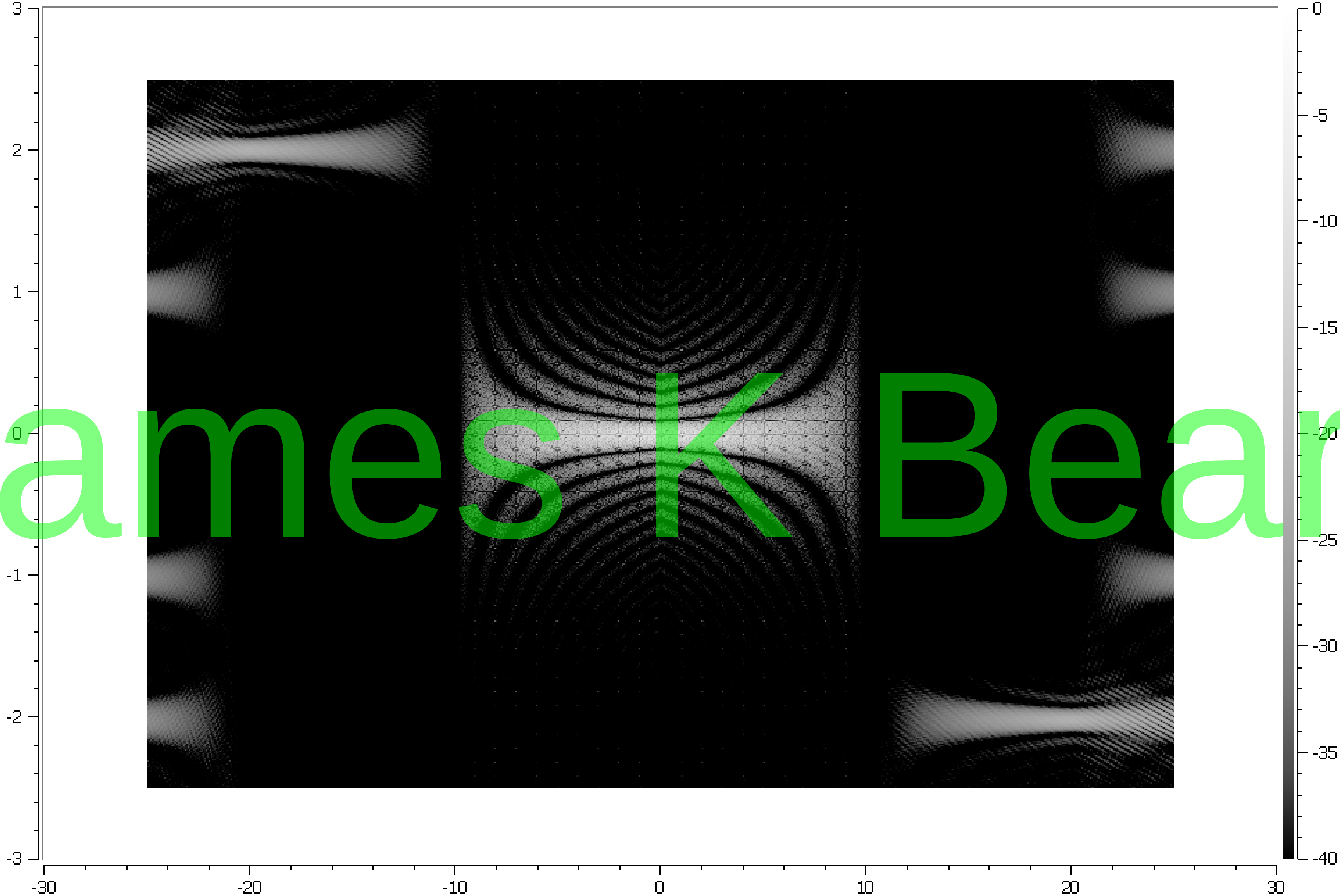
# From Another Order 14



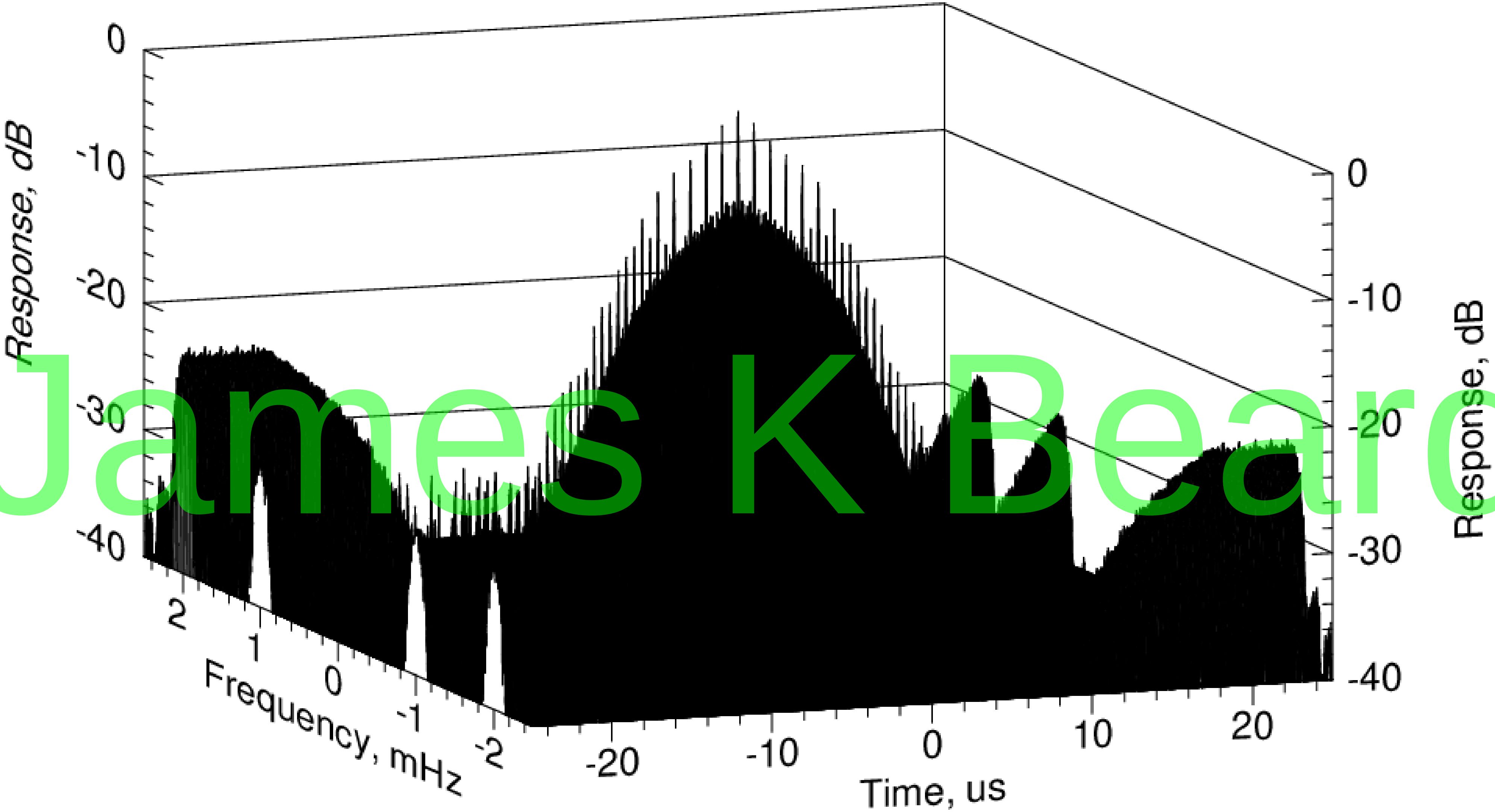
# Ambiguty Function w/Chirp



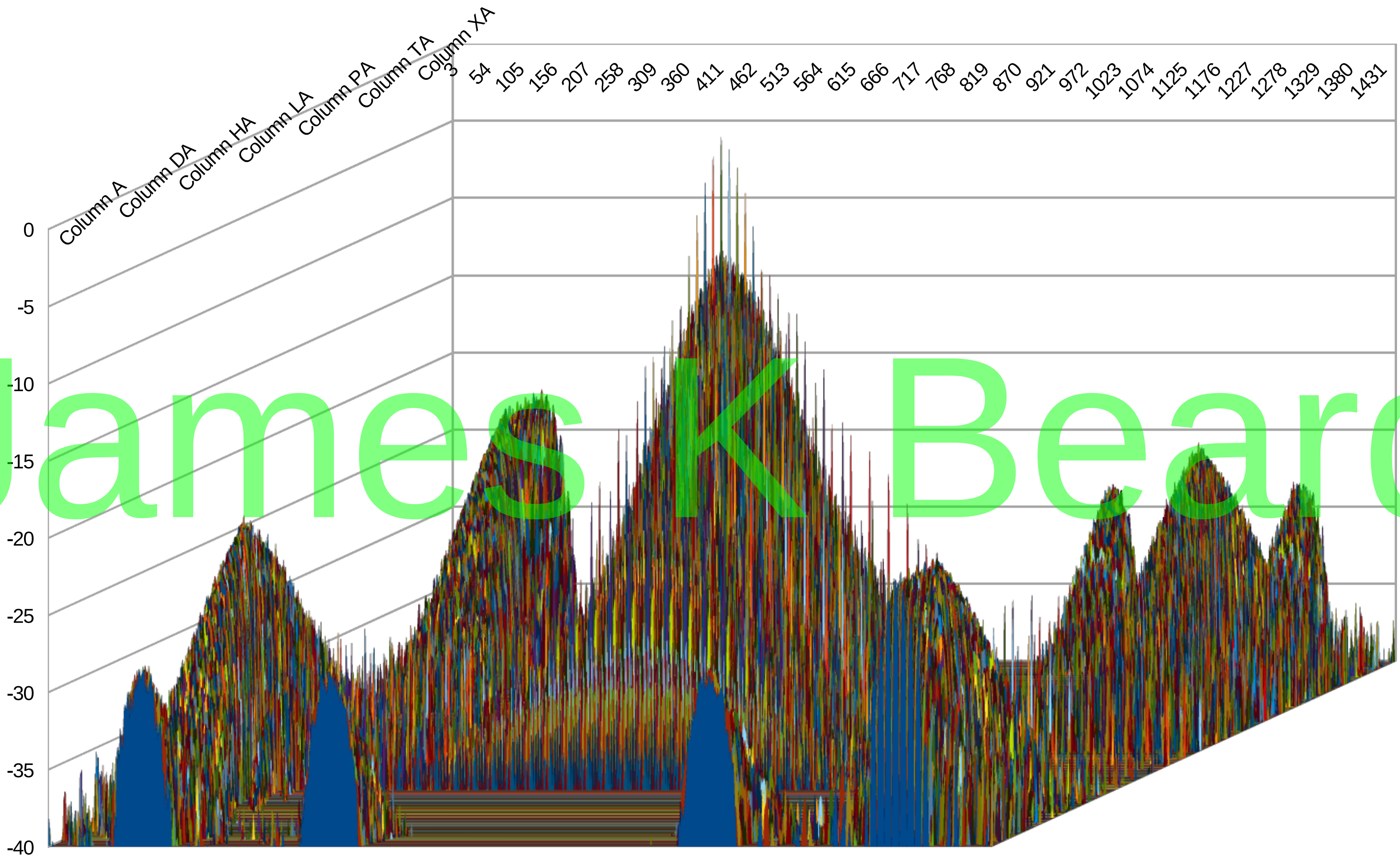
# Central 5X5 Area



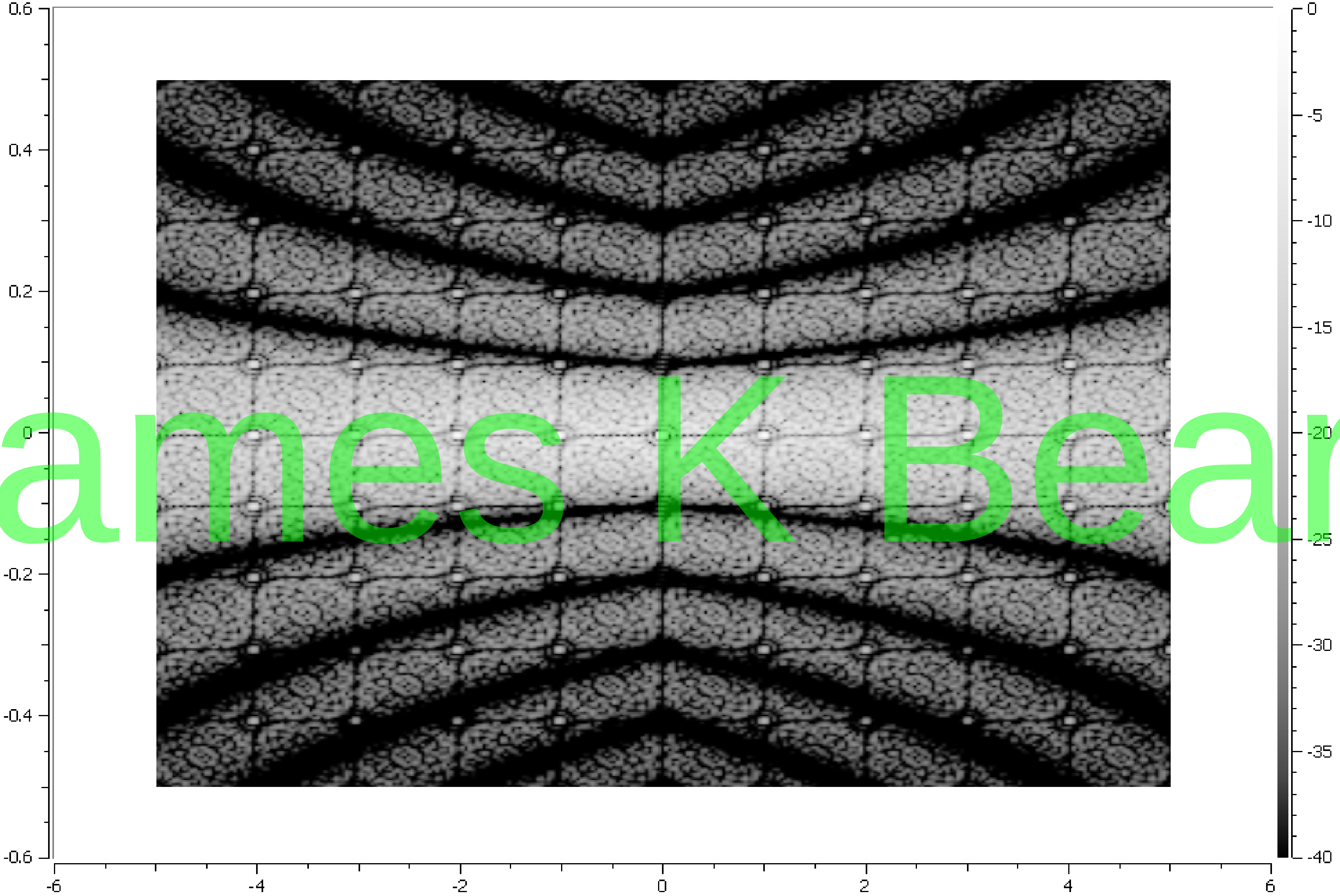
# Central 5X5 Area



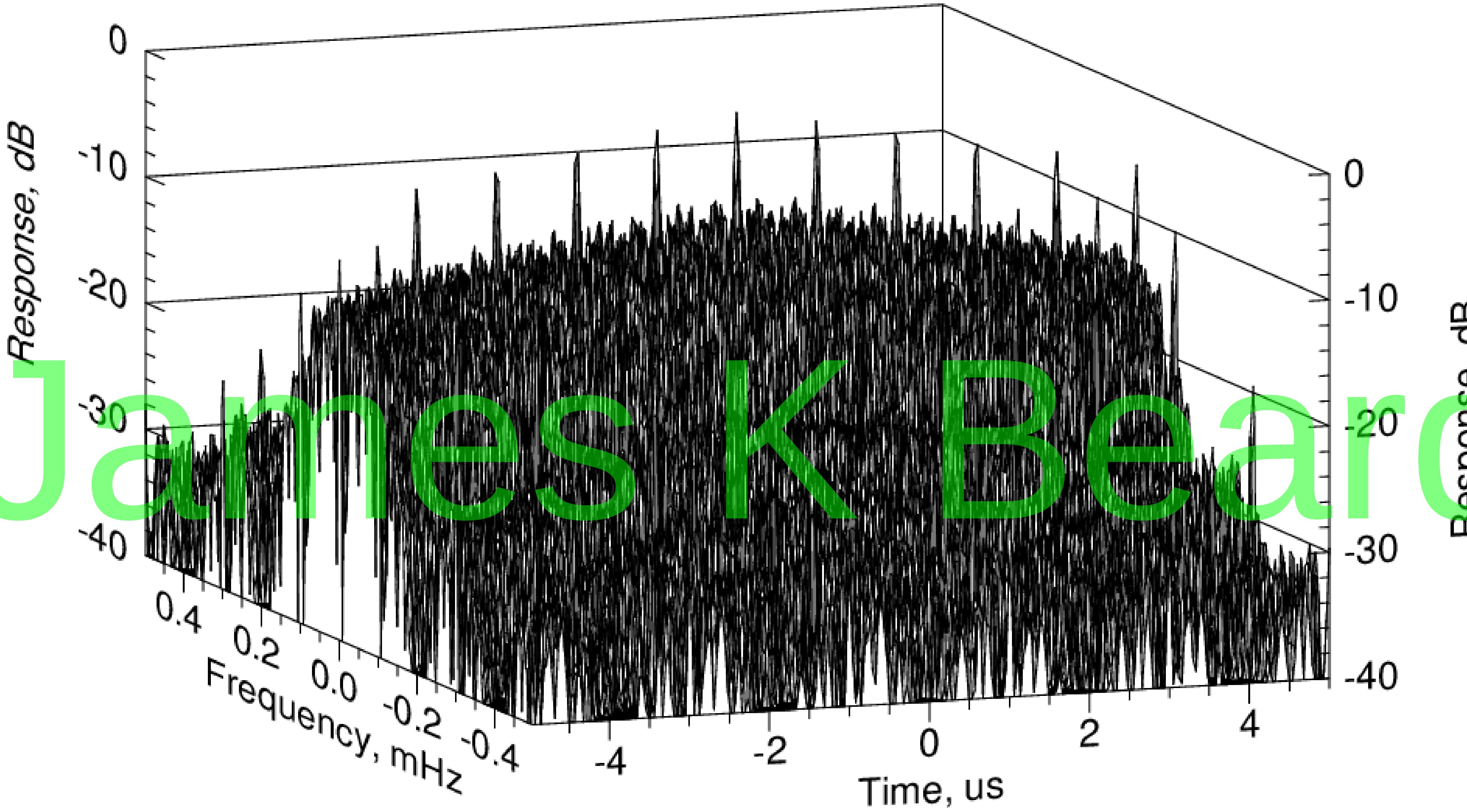
# Central 5X5 Area



# Central Square of DAF



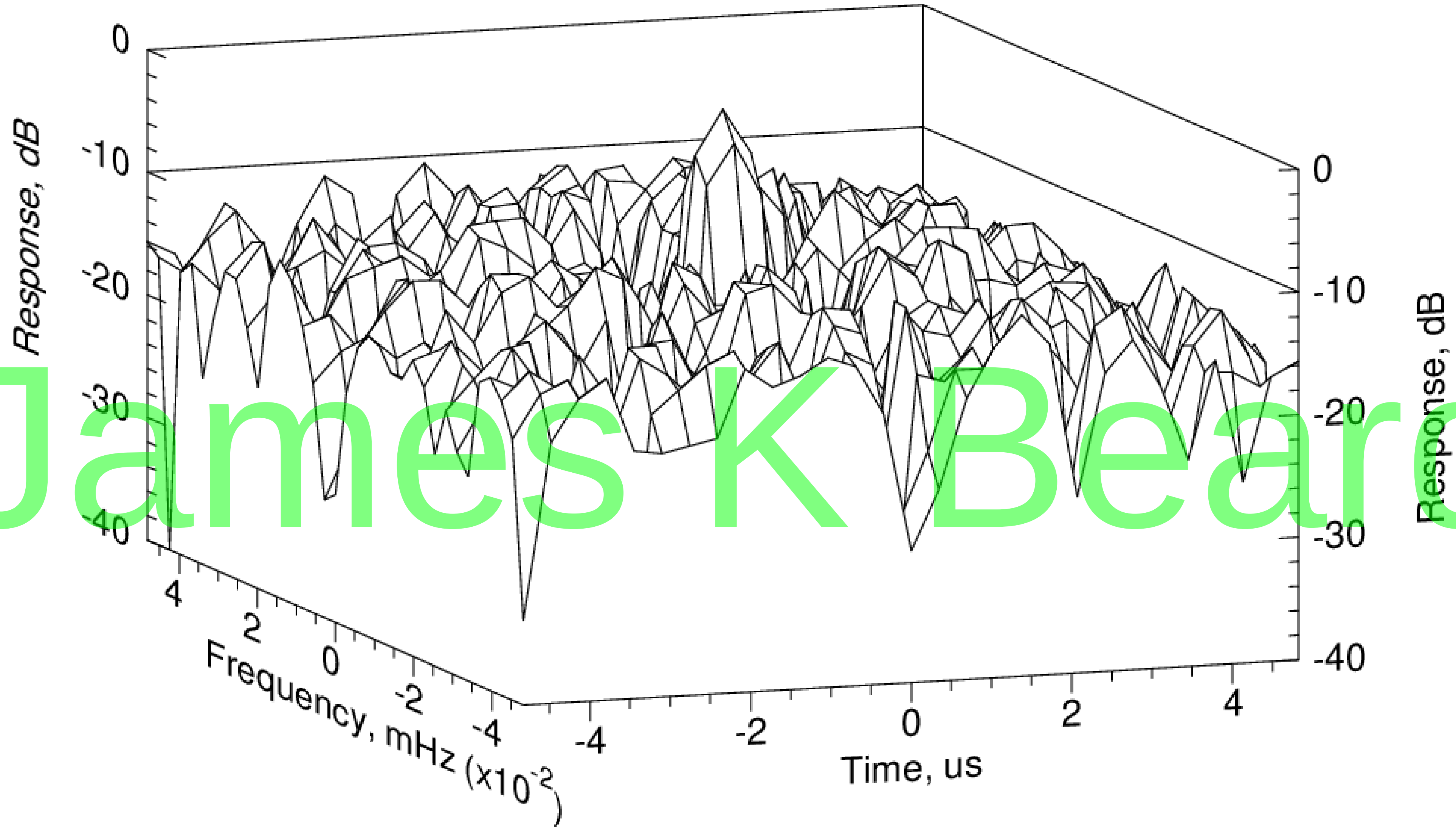
# Central Square of DAF



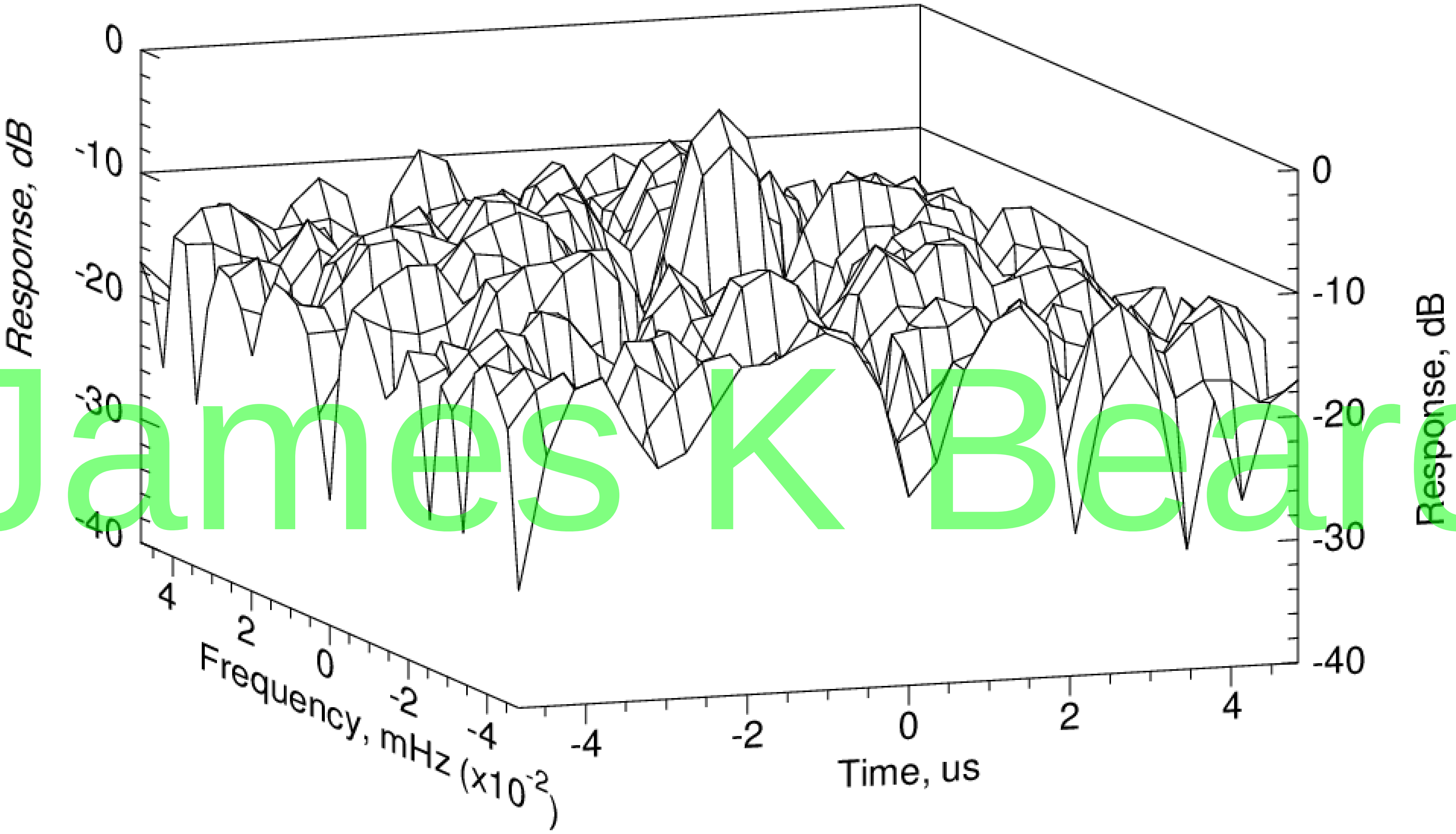
# Receiver Effects

- There is no time or frequency weighting
- Band edge rolloff in receiver is not modeled
- What happens with band edge weighting?
- For the purposes of illustration
  - Taylor weight in frequency domain
  - Bandwidth is signal bandwidth
  - Crude model of Bessel or linear phase IF filter

# Central DAF Square WO/Rolloff



# Central DAF Square W/Rolloff



# Takeaways

- Changes due to concatenating techniques
  - Expected – replicated ambiguity functions of the chirp
  - From higher correlation peak – reduced splatter sidelobes relative to central maximum peak
- Other design opportunities
  - Shade the chips – by amplitude modulating the transmitter
  - Vary the parameters – larger Costas arrays, shorter chips, etc.
  - Use other waveforms in the chips, including FSK
- We are just scratching the surface here

# Costas Arrays in the Future

- Where are they now?
  - High performance radar waveforms
  - Cell phone and other communications waveforms
  - Coding schemes in digital waveforms
  - Digital watermarking
- Where will the be next?
  - Anywhere minimal cross-correlation is important
  - Wherever math and physics opens a possibility

# Conclusions

- Costas array work appeared in volume in the 1970s and early 1980s
- Moore's Law and computer resources for researchers provided opportunities for new work into the 1990s
- Moore's Law and increasing complexity of radar and communications systems provides incentive for new work in the 2000s and 2010s

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