Selection of Costas Arrays to Minimize Cross-Interference

An Elementary Example

James K Beard
jkbeard@ieee.org

Topics (1 of 2)

- Definition and Use of Costas Arrays
- Underlying Mathematics
  - Finite Fields
  - A Few Interesting Identities
- The Costas Condition
  - The Difference Triangle
  - Matrix Form
- SVD of the Matrix Form

Topics (2 of 2)

- Cross-Correlation of Different Costas Arrays
- Selection of Costas Arrays Using the SVD
  - The Databases
  - Using Weighted Right Eigenvectors
- An Elementary Example
  - Cross-Correlation Between Simple Waveforms
  - Example of the Process
- References

Base Generators

- Welch 1; for a one at (i,j)
  - $$i = \alpha^{j+c-1} \, mod \, p$$
  - Integers α a principal element, p a prime, c an offset
  - Periodic in j and c, period (p-1)
- Lempel-Golomb 2
  - $$\alpha^{i} + \beta^{j} = 1$$
  - Alpha and Beta Principal Elements in GF(q)
  - Zero not allowed for i or j, Costas array of order q-2
  - Order q a power of a prime

Generators Use Finite Fields

Generators	Mechanism	Order
Taylor 0, Welch 0, Rickard-Welch 0	Welch 1, add dot	p, a prime
Welch 1, Taylor 1, Beard 1, Rickard-LG 1	Welch 1 is base; others LG 2, add dot	q-1, q a prime or a power of a prime
Welch 2, Lempel-Golomb 2	Welch 1, drop dot; LG 2 is base	q-2, q a prime or a power of a prime
Welch 3, Lempel-Golomb 3	Welch 3, W1, drop two dots; LG 3, LG 2, drop dot	q-3, q a prime or a power of a prime
Lempel-Golomb 4, Taylor 4, Golomb* 4	Lempel-Golomb 2, two dots removed	q-4, LG 4, $q=2^k$; T 4, q=p; G*4, q=p, some found for $q=2^k$
Golomb* 5	Lempel-Golomb 2, three dots removed	q-5, always follows G*4; q=p; some found for $q=2^k$

Welch 1 based require q=p, Lempel-Golomb based allow $q=p^k$, k>0

“Searching” Generalizations

- The Searching Generalizations
  - Not based on universally applicable logic with proof
  - Their use does not always produce a Costas array
  - They do not produce Costas arrays of order greater than 57
- The Methods
  - Add or subtract dots with *ad hoc* reasoning
  - Operate on known Costas arrays before adding dots according to a specified rationale

The Searching Generalizations

Generator	Maximum Order
Welch 0	53
Taylor 0	47
Taylor 1	52
Taylor 4	57
Beard additive 1	52
Beard subtractive 1	52
Rickard-Welch 0	53
Rickard-Lempel-Golomb 1	52

Finite Fields

- Sets of objects with commutative addition, subtraction, multiplication, division
  - A zero, which is an additive identity element
  - A one, which is a multiplicative identity element
- Galois fields, exist for orders $q=p^k$, p a prime, k a positive integer[1]
- Simplest example is arithmetic modulo a prime
- Base properties include[1]
  - $x^q=x$, so $x^{q-2}=1/x$, with multiplication defines division
  - Any set GF(q) is isomorphic to any other such set

Major Reference for Finite Fields

[1] Moore, E. H. (1896), "A doubly-infinite system of simple groups", in E. H. Moore; et al. (eds.), Mathematical Papers Read at the International Mathematics Congress Held in Connection with the World's Colombian Exposition, Macmillan & Co., pp. 208–242

http://2020ok.com/books/20/mathematical-papers-read-at-the-international-mathematical-congress-held-in-connection-with-the-world-s-columbian-exposition-chicago-1893-edited-by-the-committee-of-the-congress-e-hastings-moore-os-41520.htm

"It is necessary that all quantitative ideas should be excluded from the concept *marks* [elements of GF(q)]. Note that the signs >, < do not occur in the theory."

Vector Extensions

- Lempel-Golomb generators use GF(q), $q=p^k$, $k>0$, can use vector extensions
- Most implementations of GF(q) use Conway polynomials
  - Order k-1 polynomials characterized by k coefficients that are integers modulo p
  - Addition, subtraction, multiplication done by polynomial arithmetic, then coefficients are taken modulo p
  - Multiplication results resolved to order k-1 by taking modulo an irreducible monic generating polynomial, of order k
  - Division by any element α is multiplication by α^{q-2}

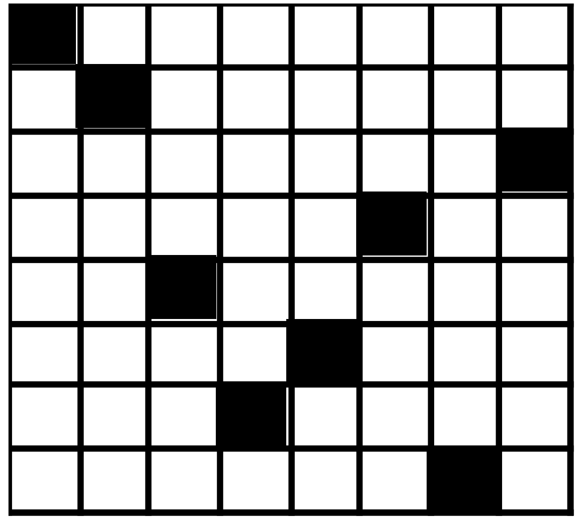
The Costas Condition

- Base definition
  - Given a simple frequency jump burst waveform and a matched filter, then, with mismatch in either or both range and Doppler (frequency), no more than one pulse in the burst will correlate in the matched filter.
- Commonly used ways of posing the Costas Condition
  - The difference triangle
  - Difference vectors
  - Discrete Ambiguity Function (DAF)
  - They are mathematically equivalent

The Difference Triangle

- Based on the Costas array vector c(j)
- Difference triangle has n-1 rows
- Each row number i has (n-i) elements
- Each row consists of differences
  - Row i, column j: $d(i,j)=c(j+i)-c(j)$
- Costas condition:
  - No two elements of any given row are equal
  - No element is zero because elements of c(j) aren't repeated

Example of Difference Triangle



Costas array:		1	2	5	7	6	4	8	3
Row	1	1	3	2	-1	-2	4	-5	
Row	2	4	5	1	-3	2	-1		
Row	3	6	4	-1	1	-3			
Row	4	5	2	3	-4				
Row	5	3	6	-2					
Row	6	7	1						
Row	7	2							

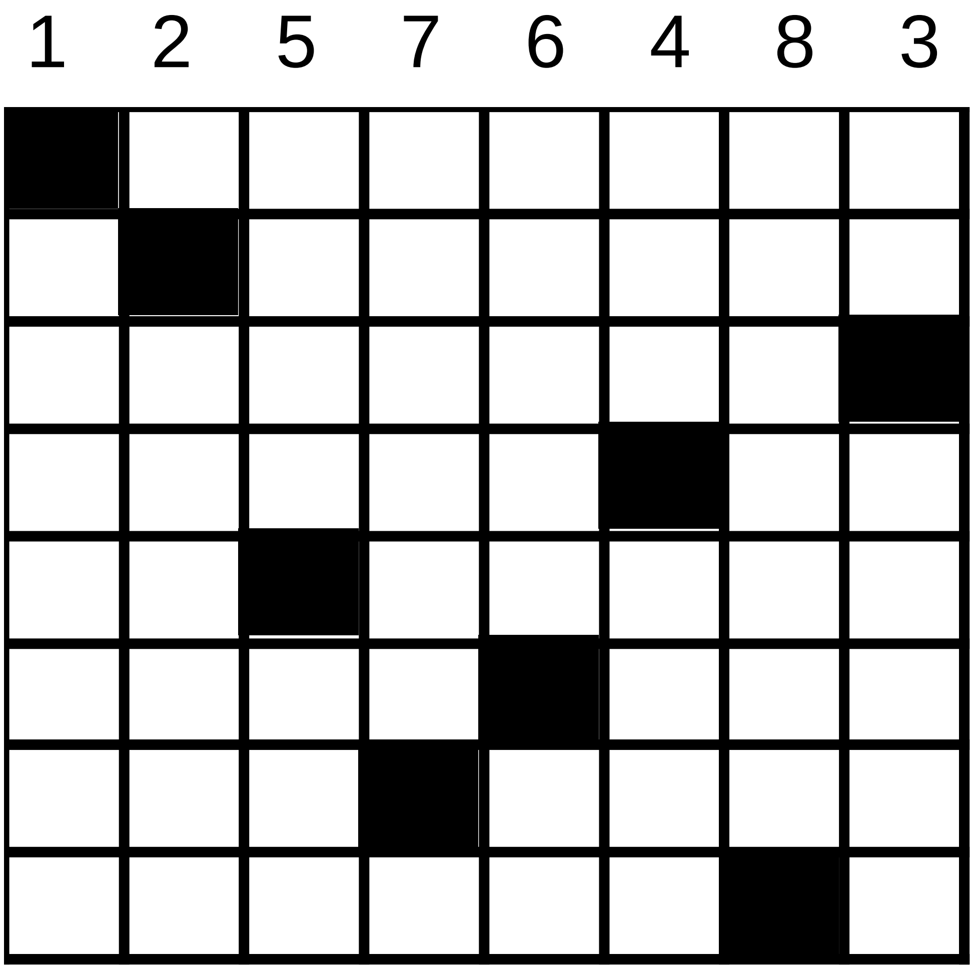
Difference Vectors

- A Difference Vector is the vector between any two ones in a Costas array matrix
- The Costas condition is that no two of them are equal
- Relationship to the Difference Triangle
  - Two ones
    - Column j, row c(j), and column (j+i), c(j+i), d(i,j)=c(j+i)-c(j)
    - Distance vector dv(i,j) = (d(i,j), i)
  - There is also the one pointing the other way, -dv(i,j)

Discrete Ambiguity Function

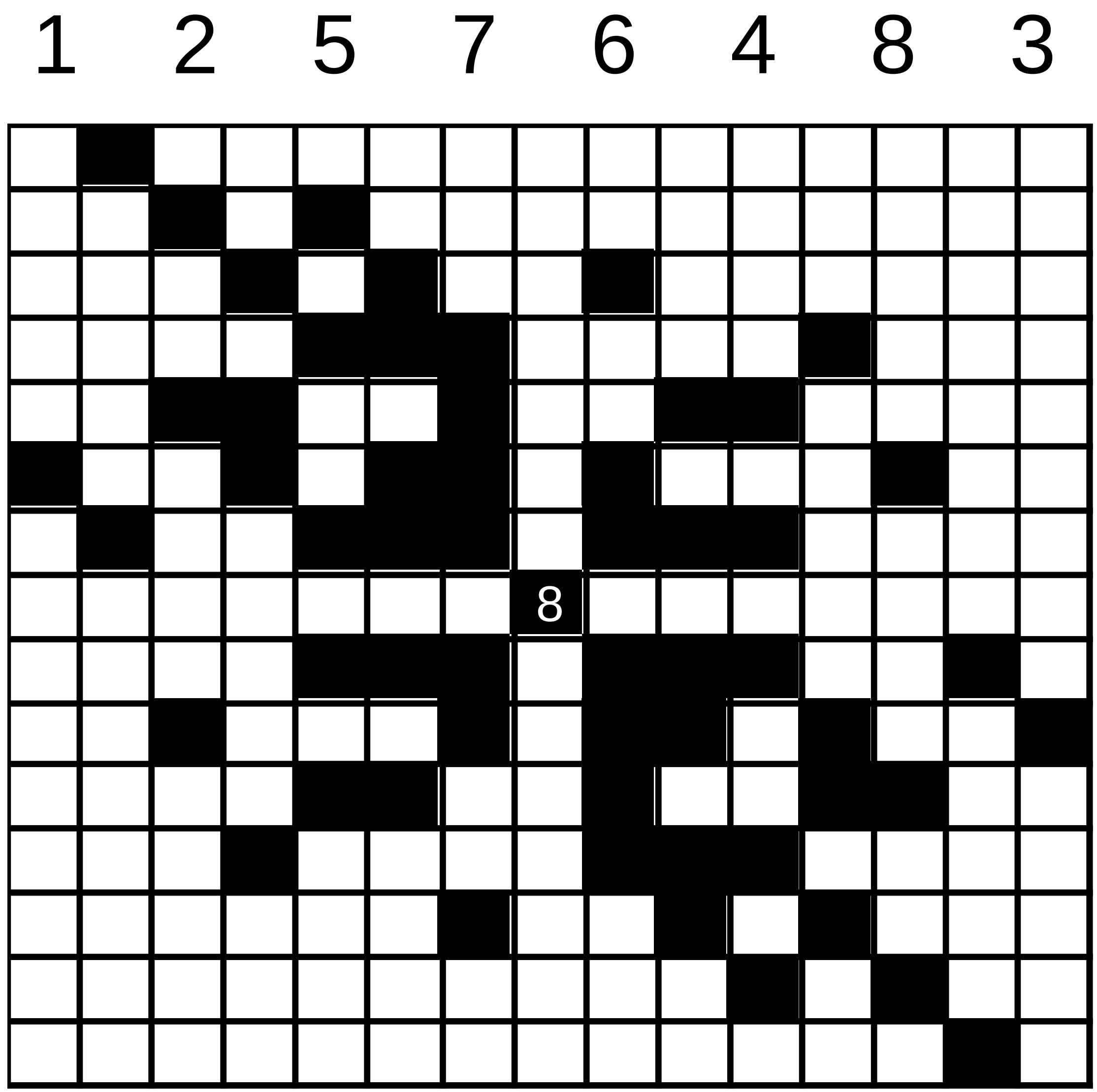
- The DAF is a simple concept related to the ambiguity function
- Defined as a cross-correlation between two Costas array matrices
- Result is a square matrix 2n-1 elements on a side
- Center element is n
- Rest is zeros except for ones at positions defined by all the distance vectors

Example of Distance Vectors



(1,1), (3,1), (2,1), (-1,1), (-2,1), (4,1), (-5,1)
(4,2), (5,2), (1,2), (-3,2), (2,2), (-1,2)
(6,3), (4,3), (-1,3), (1,3), (-3,3)
(5,4), (2,4), (3,4), (-4,4)
(3,5), (6,5), (-2,5)
(7,6), (1,6)
(2,7)

Example of DAF



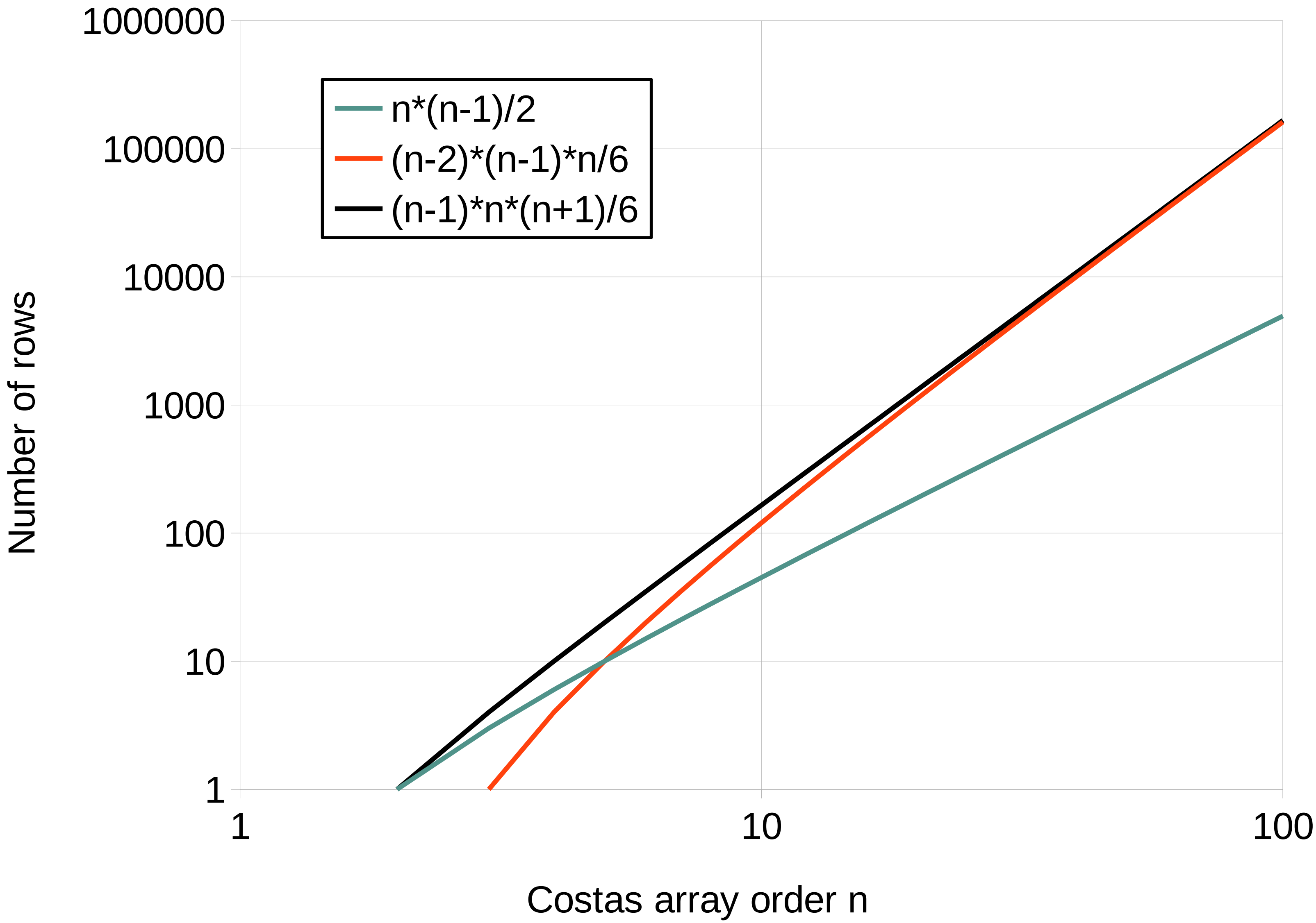
Matrix Form of the Costas Condition

- Simplest when formulated from the difference triangle
- Concept is a “tall” matrix A (many more rows than columns) times the Costas array vector as a column vector
- Result of $A \cdot \vec{c}$ is a long column of integers
- Costas condition is that no element of the result vector is zero

Definition of the Matrix

- First $n \cdot (n-1)/2$ rows
  - Elements are zeros, a +1, and a -1, elements of the result vector are elements of the difference triangle
  - Elements of the result vector are nonzero because Costas array matrices have only a single one in each row (or column), so no two elements of $c(j)$ are equal
- Next $(n-2) \cdot (n-1) \cdot n/6$ rows are differences between two elements in each of the first $n \cdot (n-1)/2$ rows
- Total total number of rows is $(n-1) \cdot n \cdot (n+1)/6$

Number of Rows of A



Singular Value Decomposition

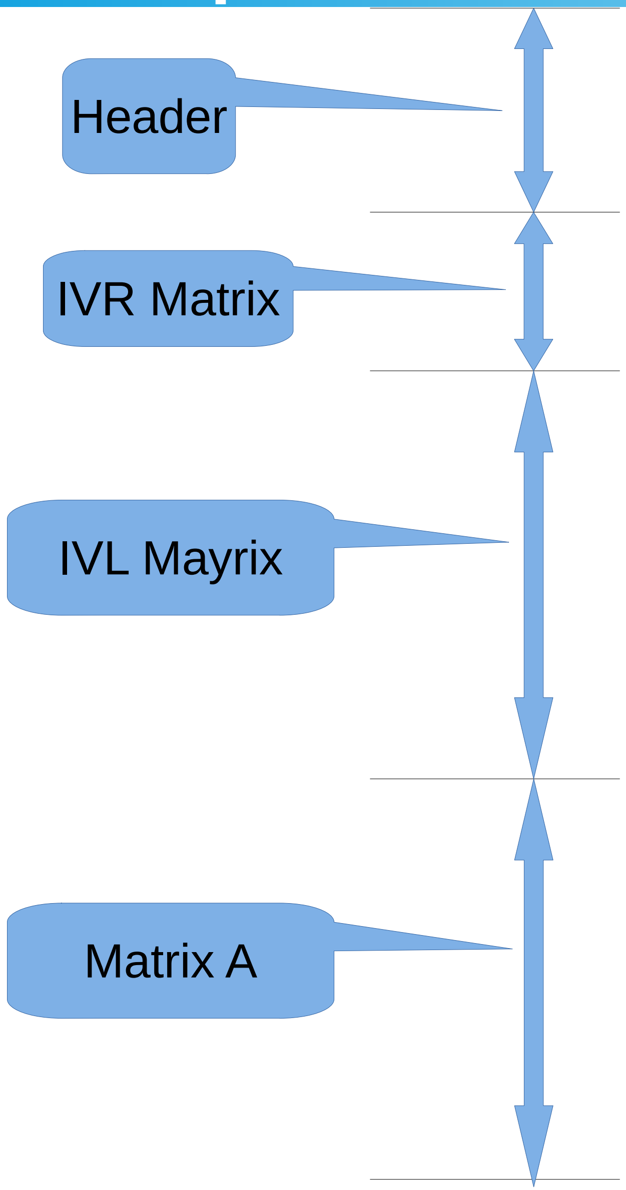
- Simple equation
  - $A = VL \cdot \Lambda \cdot VR^T$
  - VL is a matrix whose columns are left eigenvectors $\overrightarrow{vl}_i$
  - VL has the same shape as A
  - Λ is a diagonal n x n matrix of eigenvectors λ_i
  - VR is an n x n matrix of right eigenvectors $\overrightarrow{vr}_i$
- Reconstruction equation

$$A = \sum_{i=1}^{n} \overrightarrow{vl}_i \cdot \lambda_i \cdot \overrightarrow{vr}_i^T$$

Interpretation

- Multiplication by the right eigenvector matrix VR
  - Rotates the input vector
  - When VR is a left-handed coordinate system (determinant is -1), VR reflects the input vector in one dimension
  - Result is input vector rotated/reflected into “eigenspace”
- Scales the rotated vectors by λ_i
- Left eigenvector matrix VL expresses the rotated and scaled input vector in left eigenvector space

Example of SVD of A from Database



4	10	6	4	Order - total_rows - rows_order_1 - rows_order_2
4	20	4	20	Squared lengths of right eigenvectors
0	20	14	70	Squared lengths of left eigenvectors
0	4	14	14	Squared eigenvalues
0	10	2	10	Scale factors
1	-3	-1	1	Row 1 of right eigenvector matrix IV
1	-1	1	-3	Row 2 of right eigenvector matrix IV
1	1	1	3	Row 3 of right eigenvector matrix IV
1	3	-1	-1	Row 4 of right eigenvector matrix IV
0	1	1	-2	Row 1 of left eigenvector matrix IVL
0	2	1	1	Row 2 of left eigenvector matrix IVL
0	3	0	-1	Row 3 of left eigenvector matrix IVL
0	1	0	3	Row 4 of left eigenvector matrix IVL
0	2	-1	1	Row 5 of left eigenvector matrix IVL
0	1	-1	-2	Row 6 of left eigenvector matrix IVL
0	0	1	-5	Row 7 of left eigenvector matrix IVL
0	0	2	0	Row 8 of left eigenvector matrix IVL
0	0	2	0	Row 9 of left eigenvector matrix IVL
0	0	1	5	Row 10 of left eigenvector matrix IVL
-1	1	0	0	Row 1 of Costas constraint matrix A
-1	0	1	0	Row 2 of Costas constraint matrix A
-1	0	0	1	Row 3 of Costas constraint matrix A
0	-1	1	0	Row 4 of Costas constraint matrix A
0	-1	0	1	Row 5 of Costas constraint matrix A
0	0	-1	1	Row 6 of Costas constraint matrix A
-1	2	-1	0	Row 7 of Costas constraint matrix A
-1	1	1	-1	Row 8 of Costas constraint matrix A
-1	1	1	-1	Row 9 of Costas constraint matrix A
0	-1	2	-1	Row 10 of Costas constraint matrix A

Selection of Costas Arrays

- An elementary example
  - Waveform is simple FJB
  - Platforms are multiple similar systems
    - Cell phones in contact with the same tower
    - Radars in same theater of operations
    - Communications in overlapping or shared bands
- Goals
  - Eliminate crosstalk
  - Minimize cross-interference

Approach

- Use Costas arrays as frequency hopping plan for FJB waveforms
  - Use a different Costas array for each platform
  - Select Costas arrays that share as few distance vectors as possible
- General notes
  - The time-bandwidth product of a simple FJB waveform using a Costas array is n^2 times the TW product of the “chip”
  - We will look at low to moderate n

Resources

- Costas arrays
  - IEEE DataPort
  - Costas arrays and enumeration to order 1030
  - DOI 10.21227/H21P42
- Eigenvalues and right eigenvectors
  - IEEE DataPort
  - Database of Singular Value Decompositions of Matrix Representations of the Costas Condition
  - DOI 10.21227/h498-px29
- Both databases generated and posted by James K Beard
- Access through https://doi.org/<DOI>

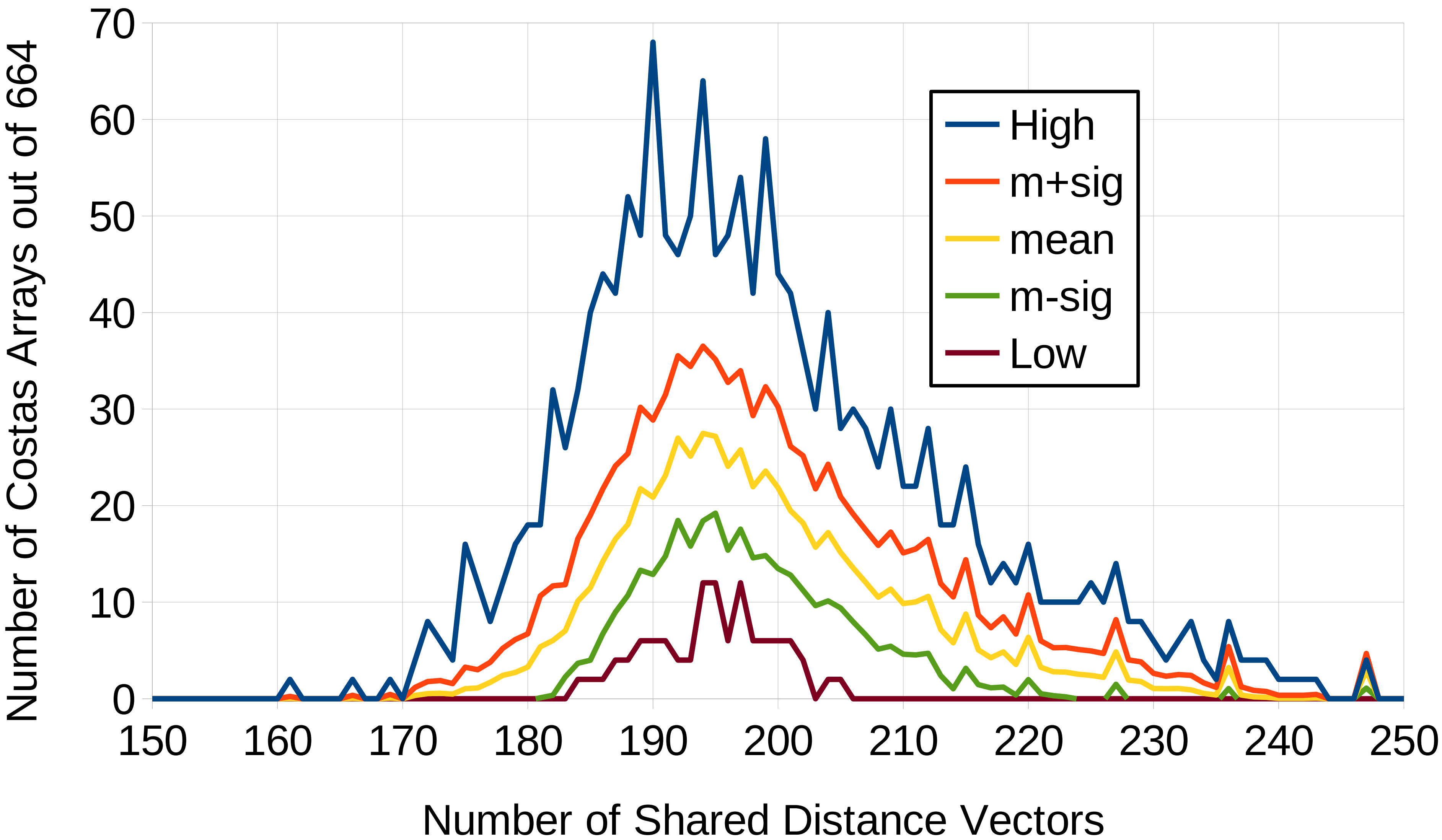
We Look at Distance Vectors

- Conceptual model
  - A waveform using one Costas array
  - A receiver matched filter using another Costas array
- For each shared distance vector
  - Two tones at a time will correlate in the matched filter
  - Single tone correlations are universal and ignored here
  - Since distance vectors are not repeated in any Costas array, more than one shared distance vector for a given offset and time and frequency cannot occur

Minimizing Shared Distance Vectors

- Basic principles
  - Other signals that don’t correlate effectively form a noise floor
  - Peaks in cross-correlation cause spikes in the noise floor
  - Shared distance vectors are a simple handle on cross-corrlelations
- A First Cut Approach
  - Use database of all known Costas arrays of a given order
  - Plot the number of Costas arrays that has a given number of cross-correlations with all the rest

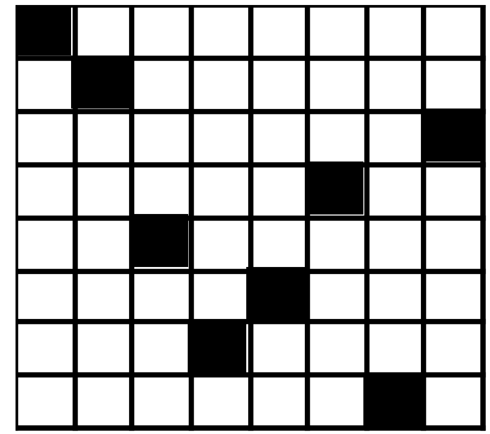
Order 30



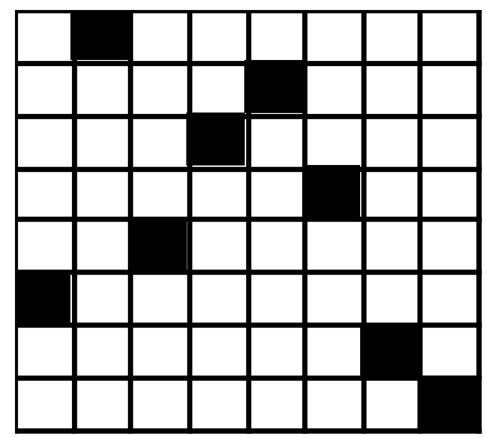
70
60
50
40
30
20
10
0
150
160
170
180
190
200
210
220
230
240
250
Number of Costas Arrays out of 664
Number of Shared Distance Vectors
High
m+sig
mean
m-sig
Low

Peaks on Distributions (1 of 2)

Costas array:		1	2	5	7	6	4	8	3
Row	1	1	3	2	-1	-2	4	-5	
Row	2	4	5	1	-3	2	-1		
Row	3	6	4	-1	1	-3			
Row	4	5	2	3	-4				
Row	5	3	6	-2					
Row	6	7	1						
Row	7	2							



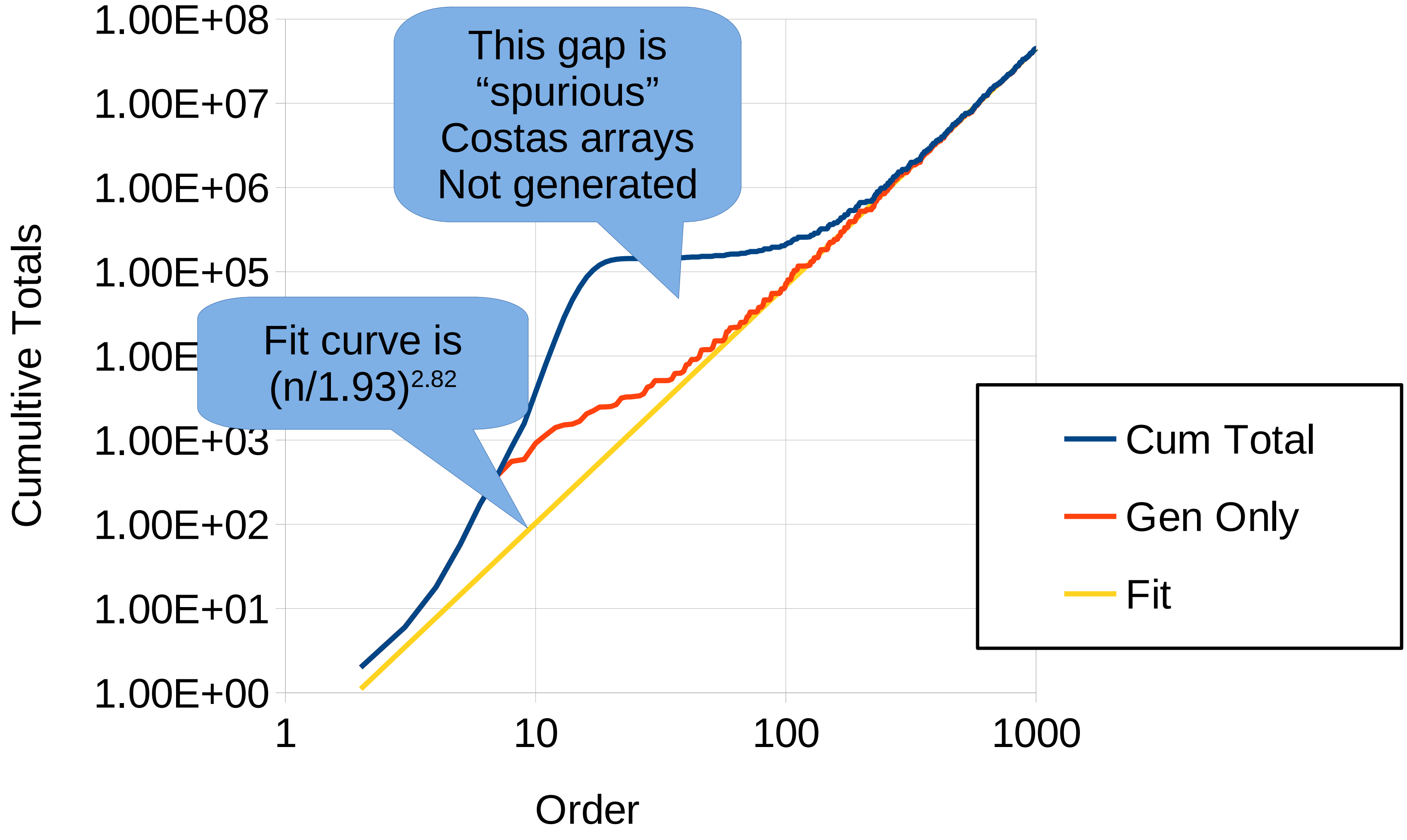
Costas array:		6	1	5	3	2	4	7	8
Row	1	-5	4	-2	-1	2	3	1	
Row	2	-1	2	-3	1	5	4		
Row	3	-3	1	-1	4	6			
Row	4	-4	3	2	5				
Row	5	-2	6	3					
Row	6	1	7						
Row	7	2							



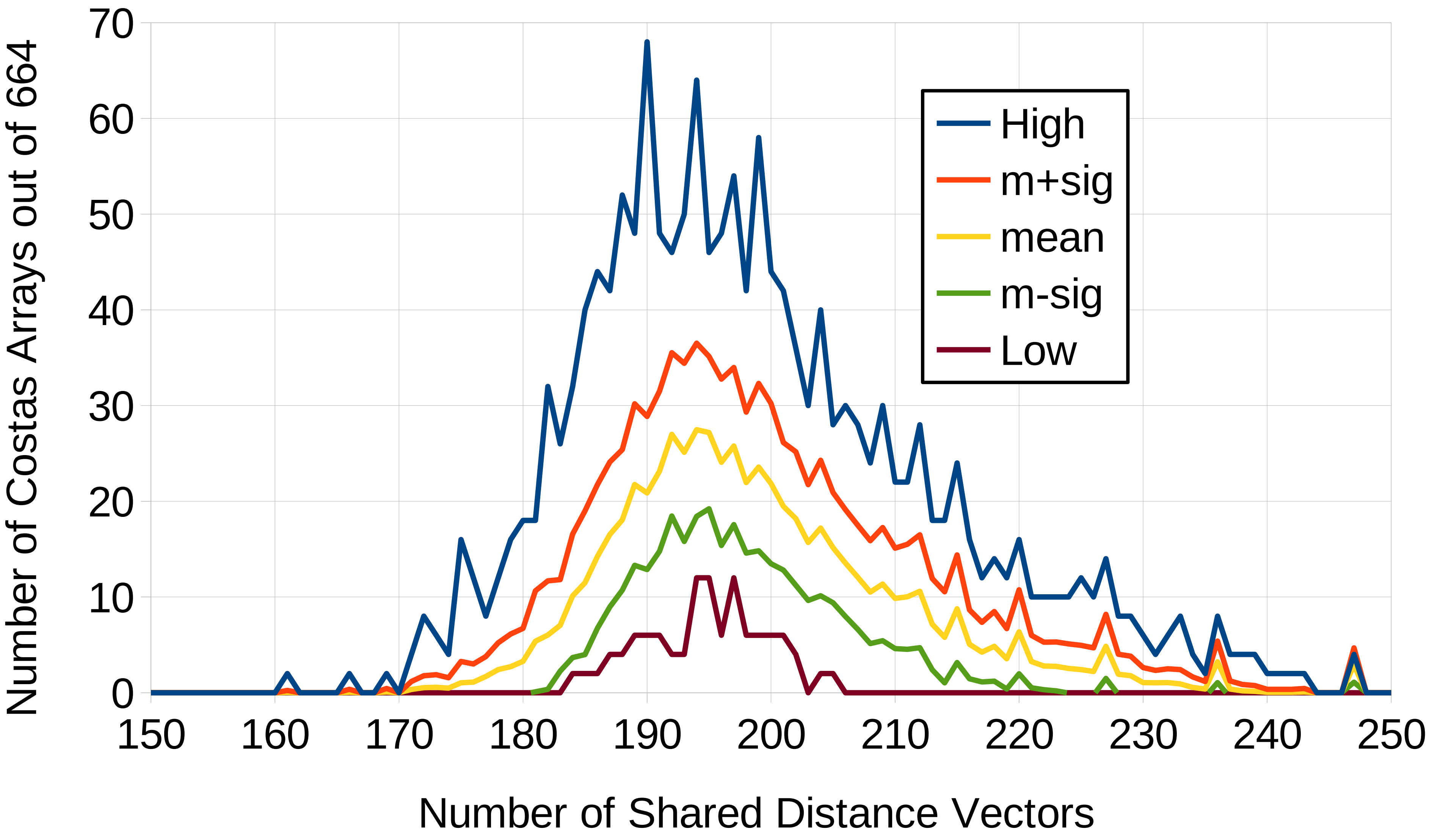
Peaks on Distributions (2 of 2)

- These two fully-correlated Costas arrays are related
  - Both rows and columns are reversed
  - Costas array matrix is rotated 180º
  - If Lempel-Golomb or extension,
    - From the same two principal elements, except that
    - The principal element.s are replaced by their reciprocals
  - If Welch or extension,
    - The offset c is increased or decreased by (q-1)/2
    - The principal element α is replace by its reciprocal α^{q-2}
- Rows of Difference Triangle
  - Same numbers
  - Reversed order
- All 28 distance vectors are shared

Costas Array Numbers vs. Order

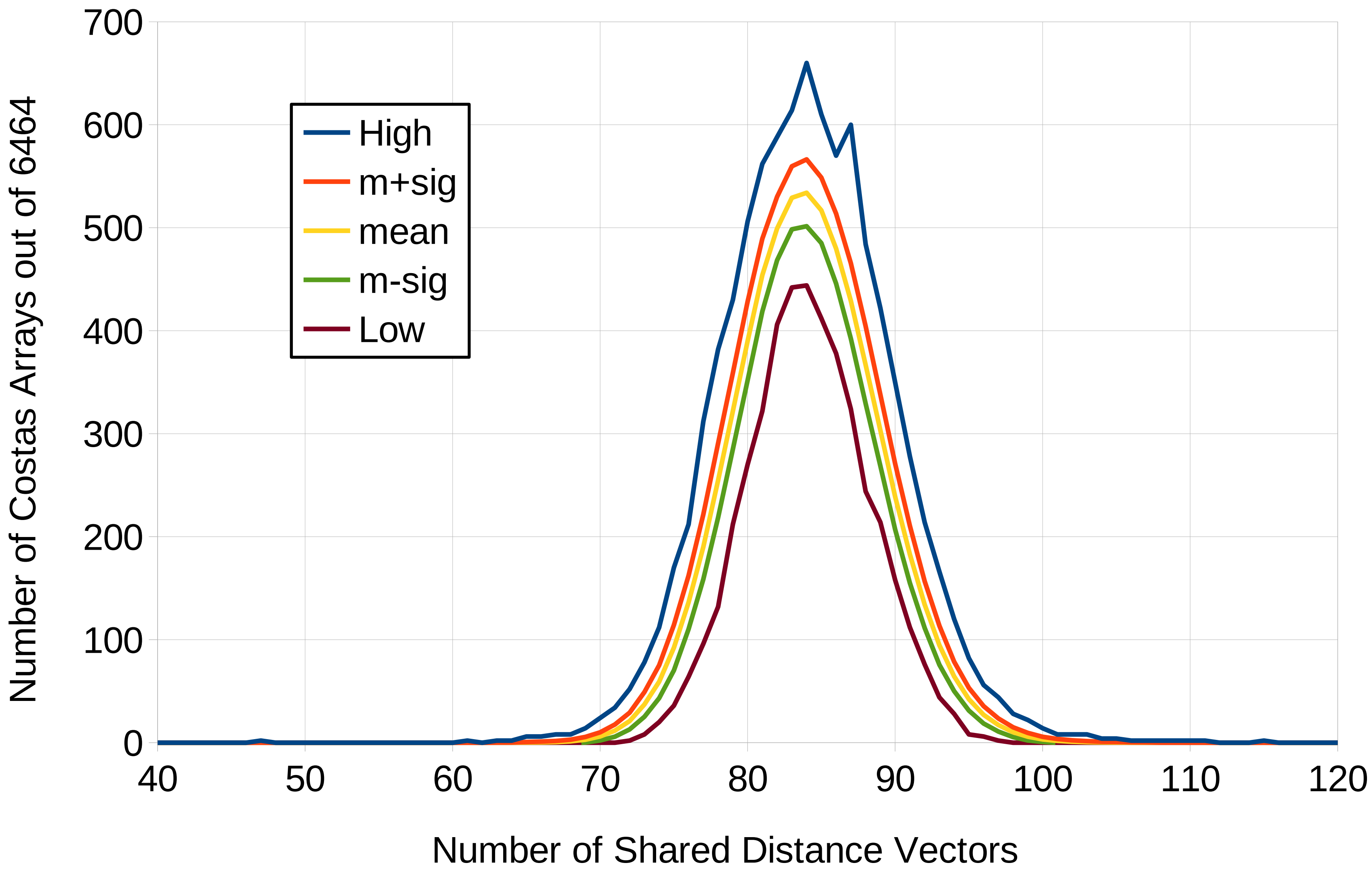


Order 30



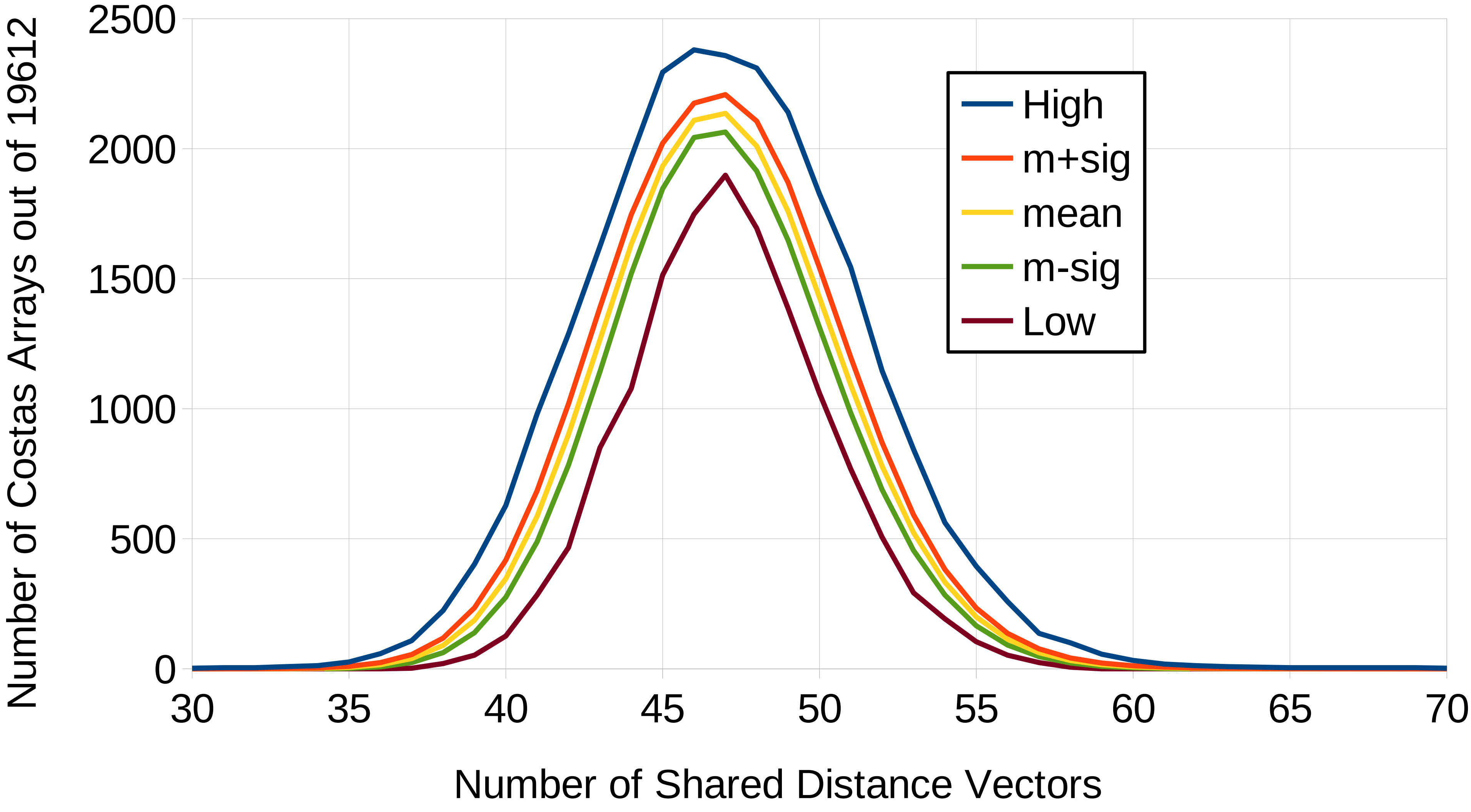
Number of Costas Arrays out of 664
Number of Shared Distance Vectors
70
60
50
40
30
20
10
0
150
160
170
180
190
200
210
220
230
240
250
High
m+sig
mean
m-sig
Low

Order 20



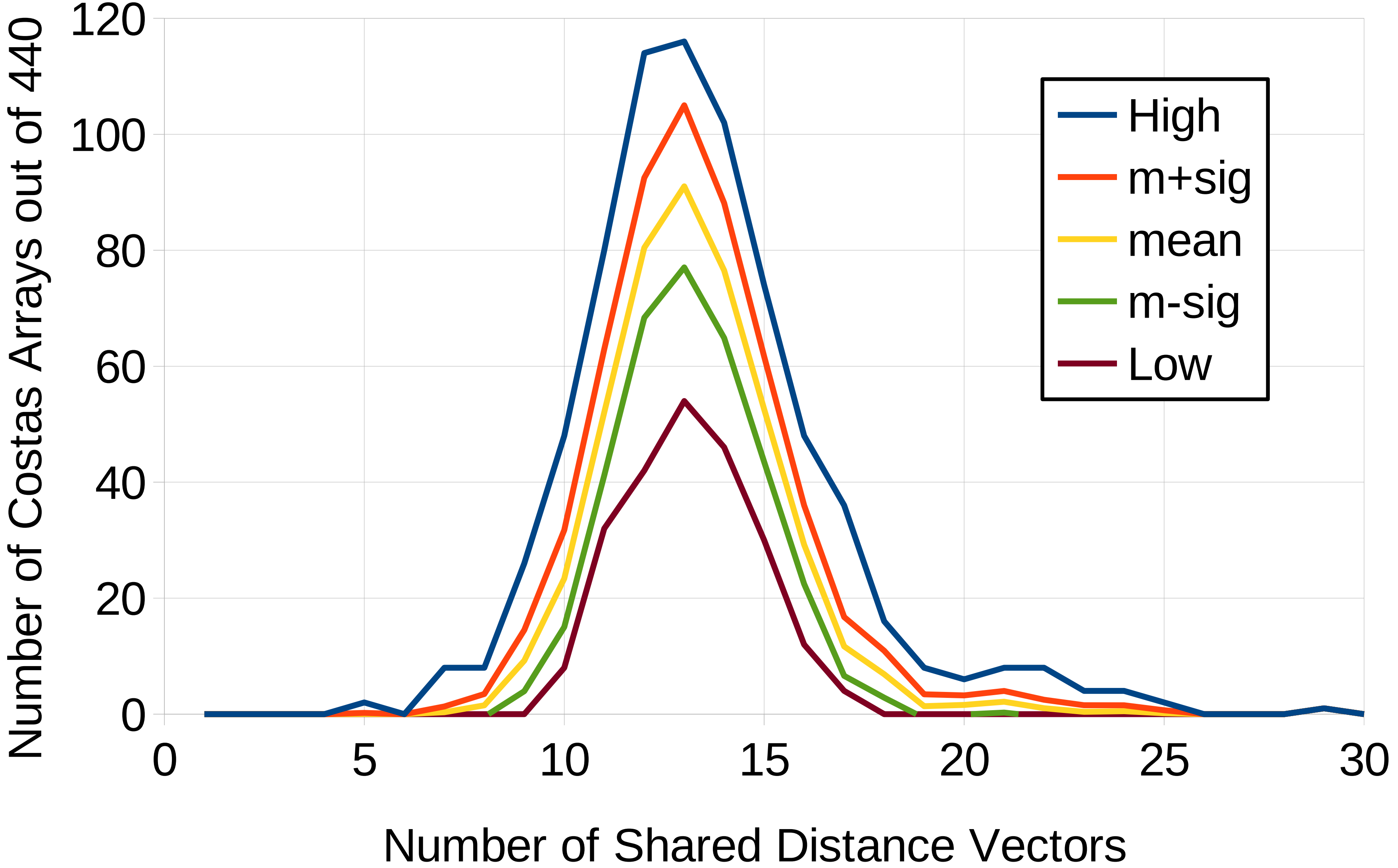
700
600
500
400
300
200
100
0
40
50
60
70
80
90
100
110
120
Number of Costas Arrays out of 6464
Number of Shared Distance Vectors
High
m+sig
mean
m-sig
Low

Order 15



Number of Costas Arrays out of 19612
2500
2000
1500
1000
500
0
30
35
40
45
50
55
60
65
70
Number of Shared Distance Vectors
High
m+sig
mean
m-sig
Low

Order 8

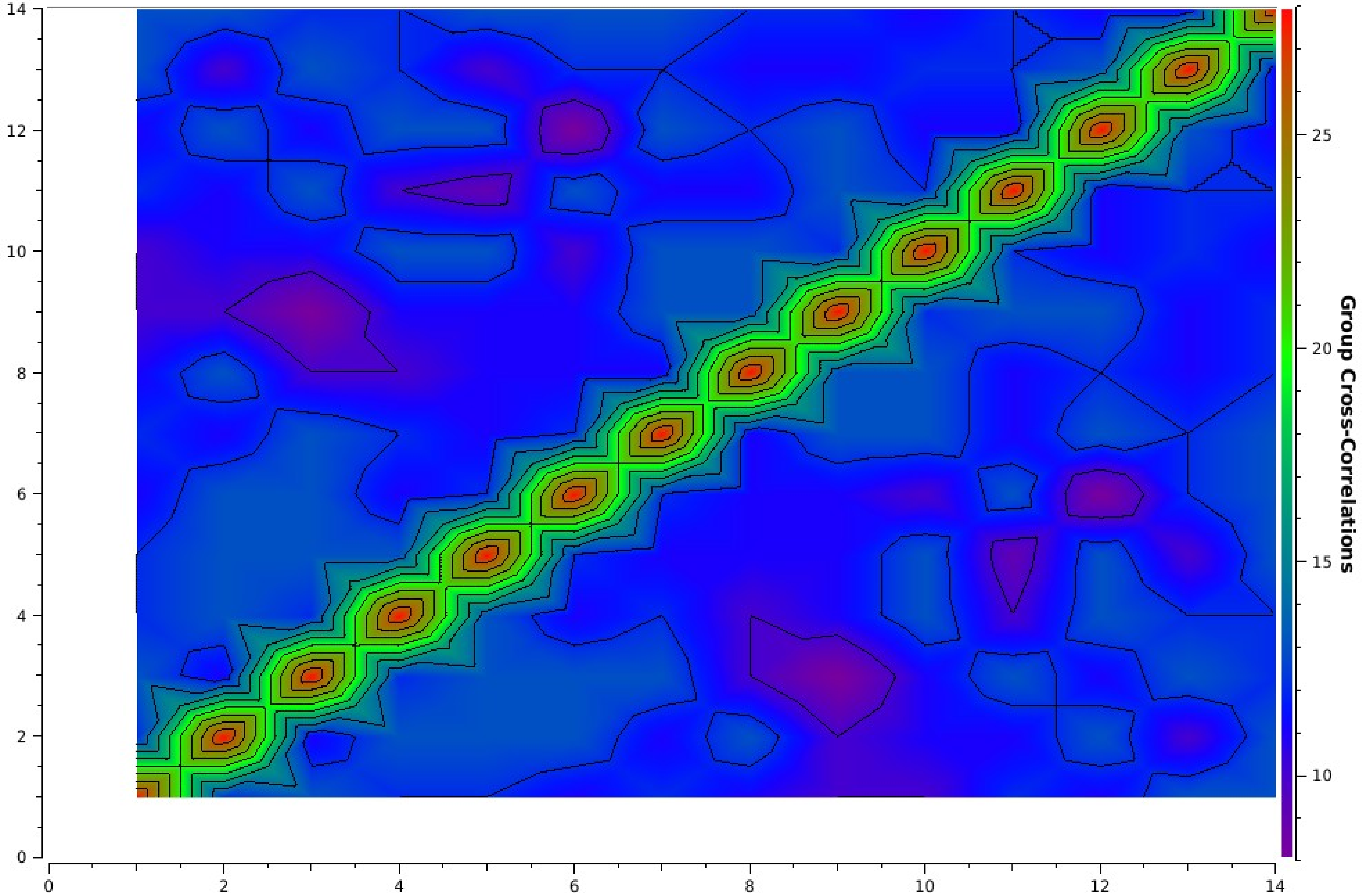


120
100
80
60
40
20
0
Number of Costas Arrays out of 440
0
5
10
15
20
25
30
Number of Shared Distance Vectors
High
m+sig
mean
m-sig
Low

A Subgroup of Interest

- Begin with a Costas array that has less than a specified threshold of p shared distance vectors with any other Costas array
- Select Costas arrays that have minimum correlation with this Costas array
- Delete members of the resulting group that have excessive shared distance vectors with other Costas arrays of the same order

Order 8, Max Correlation 13

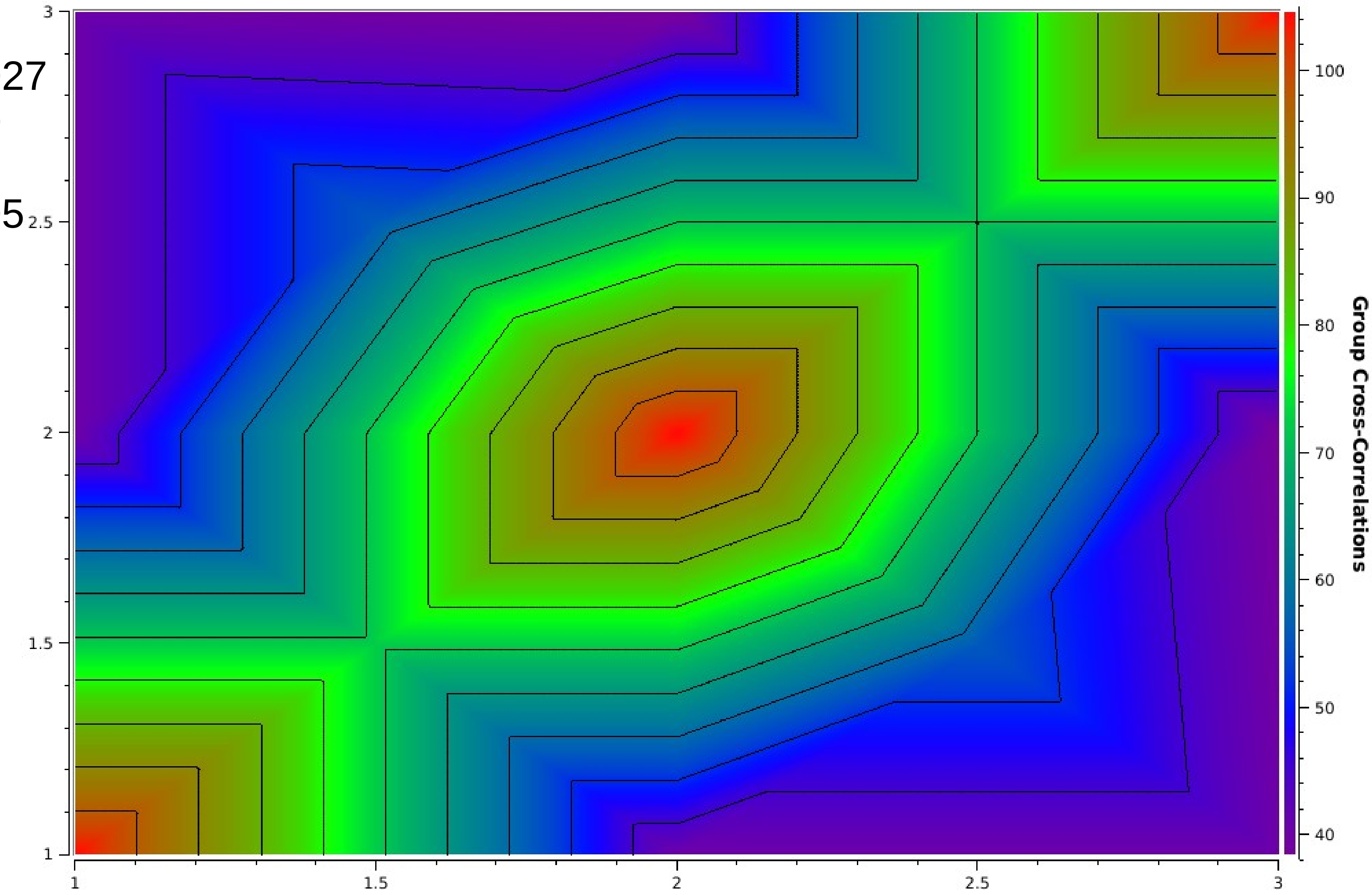


Raw Data

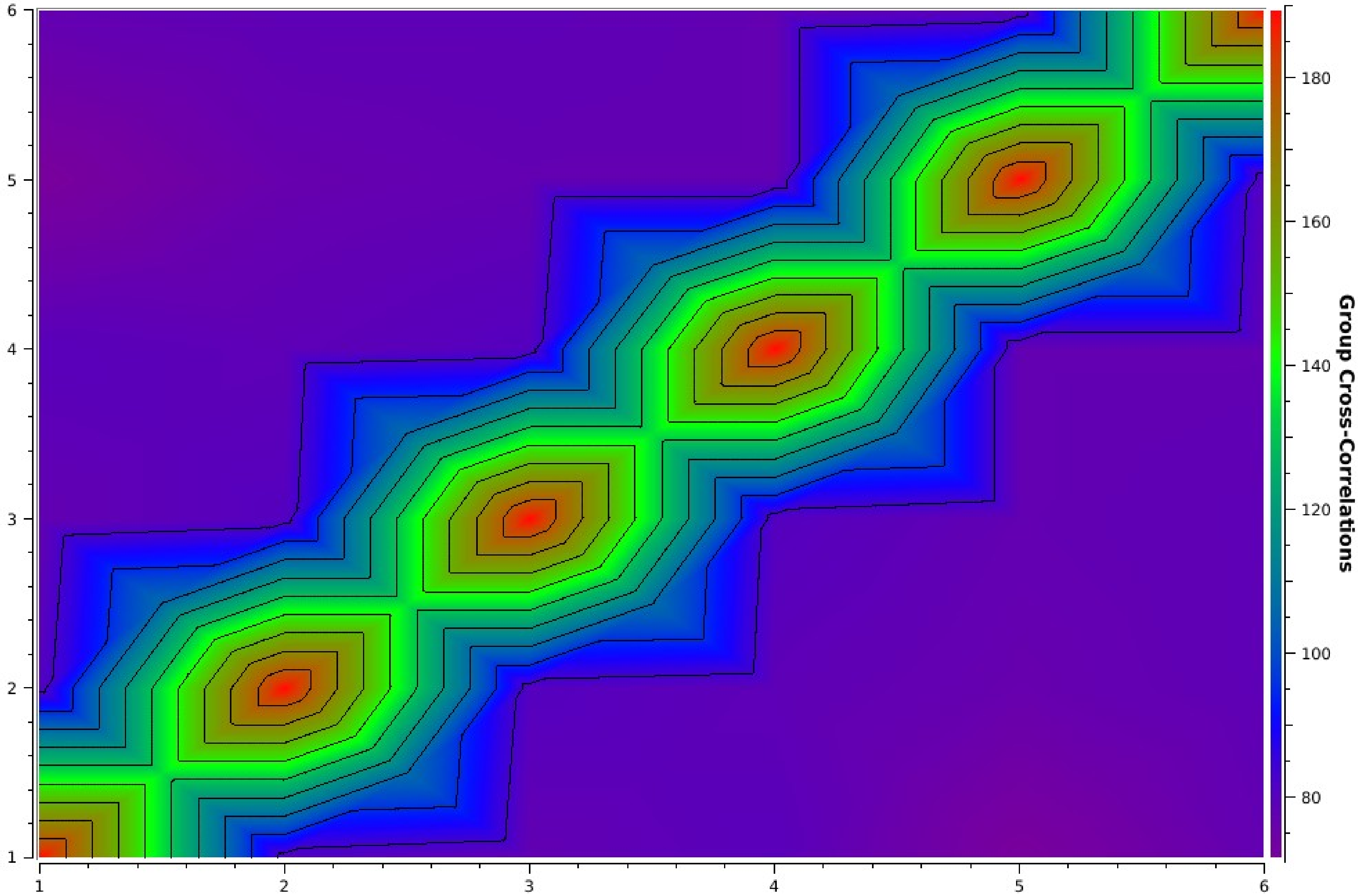
	1	6	9	17	19	27	29	69	91	116	167	171	304	392
1	28	12	13	12	12	11	12	11	10	10	12	11	13	13
6	12	28	11	13	13	13	11	13	10	11	11	13	10	13
9	13	11	28	12	13	13	13	10	8	11	13	11	13	12
17	12	13	12	28	13	11	12	10	11	13	10	13	12	12
19	12	13	13	13	28	12	11	11	11	13	9	13	10	13
27	11	13	13	11	12	28	12	11	11	10	13	8	12	13
29	12	11	13	12	11	12	28	11	13	13	11	13	12	13
69	11	13	10	10	11	11	11	28	13	13	11	12	11	12
91	10	10	8	11	11	11	13	13	28	12	13	13	11	12
116	10	11	11	13	13	10	13	13	12	28	12	11	12	11
167	12	11	13	10	9	13	11	11	13	12	28	11	12	12
171	11	13	11	13	13	8	13	12	13	11	11	28	13	11
304	13	10	13	12	10	12	12	11	11	12	12	13	28	11
392	13	13	12	12	13	13	13	12	12	11	12	11	11	28

Order 15, Max Correlations 40

	1	32	1027
1	105	40	40
32	40	105	38
1027	40	38	105



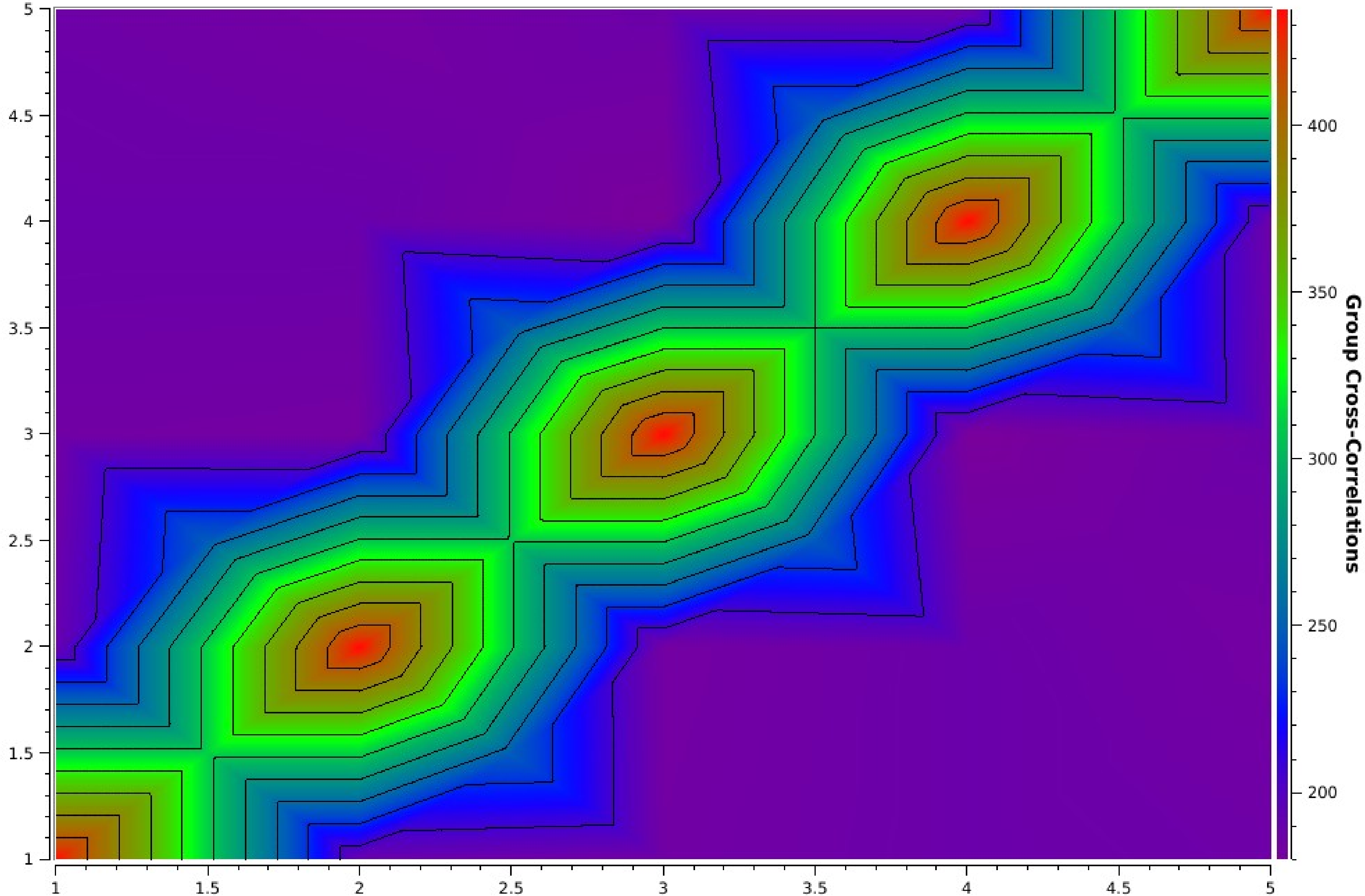
Order 20, Max Correlations 80



Order 20, Max Correlations 80

	1	4	75	181	1167	1621
1	190	80	77	79	71	77
4	80	190	80	78	76	78
75	77	80	190	80	77	78
181	79	78	80	190	77	77
1167	71	76	77	77	190	79
1621	77	78	78	77	79	190

Order 30, Max Correlations 190



Order 30, Max Correlation 190

	1	7	27	81	524
1	435	190	185	189	187
7	190	435	184	189	184
27	185	184	435	180	188
81	189	189	180	435	187
524	187	184	188	187	435

Mapping Costas Array Vectors

- Mapping is $\overrightarrow{cm_i} = \Lambda^r \cdot VR^T \cdot \overrightarrow{c_i}$
- Statistics are for
  - Entire database for that order
  - Designated group with limited cross-correlation
- Statistics:
  - Average, or reference vector
  - Mean distance from reference vector
  - RMS distance from reference vector

Equations for the Statistics

- Mean, or reference vector

$$\overrightarrow{cref} = \frac{1}{n} \cdot \sum_{i=1}^{n} \overrightarrow{cm_i}$$

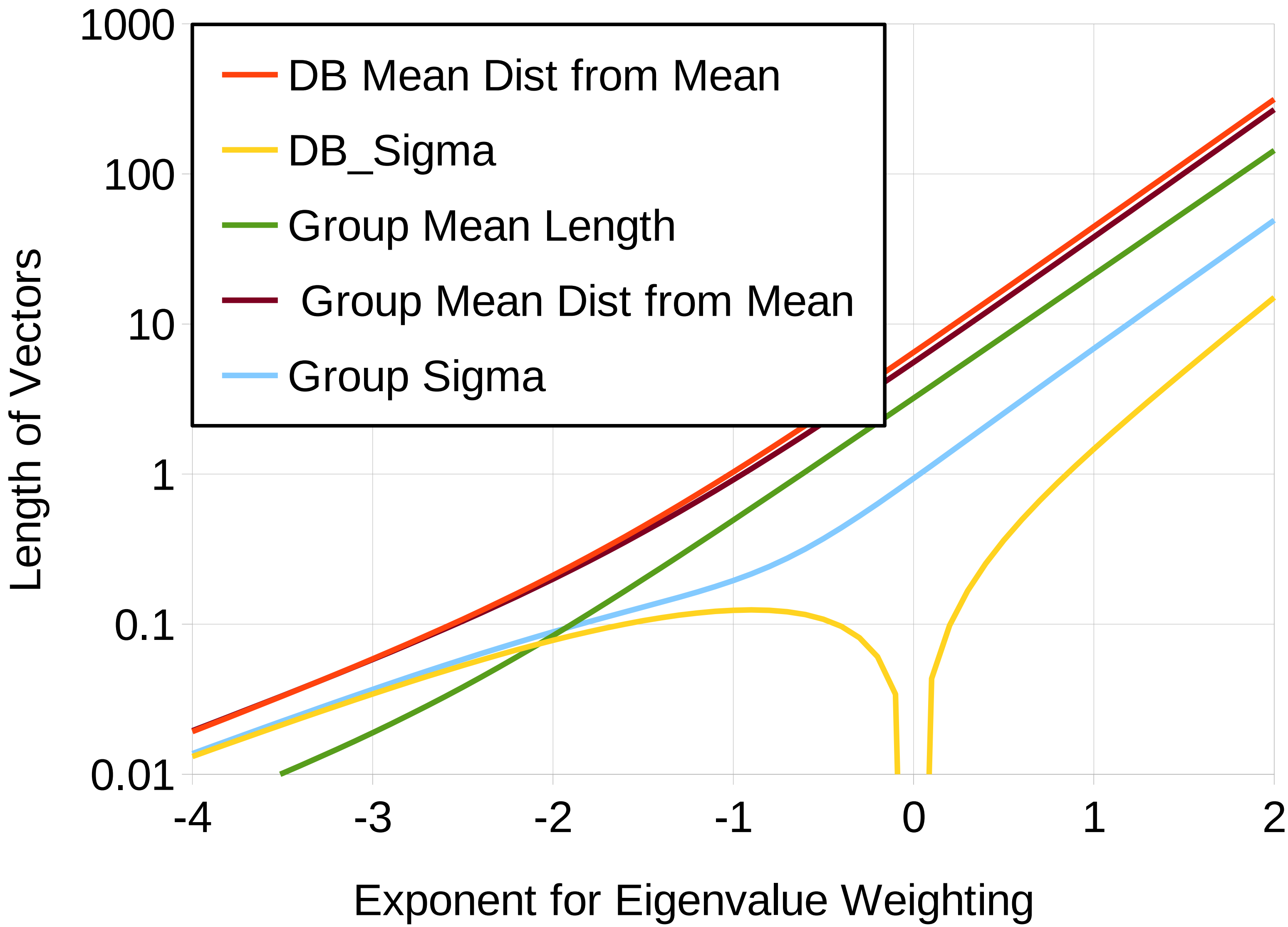
- Mean distance from reference vector

$$\bar{d} = \frac{1}{n} \cdot \sum_{1}^{n} \left| \overrightarrow{cm_i} - \overrightarrow{cref} \right|$$

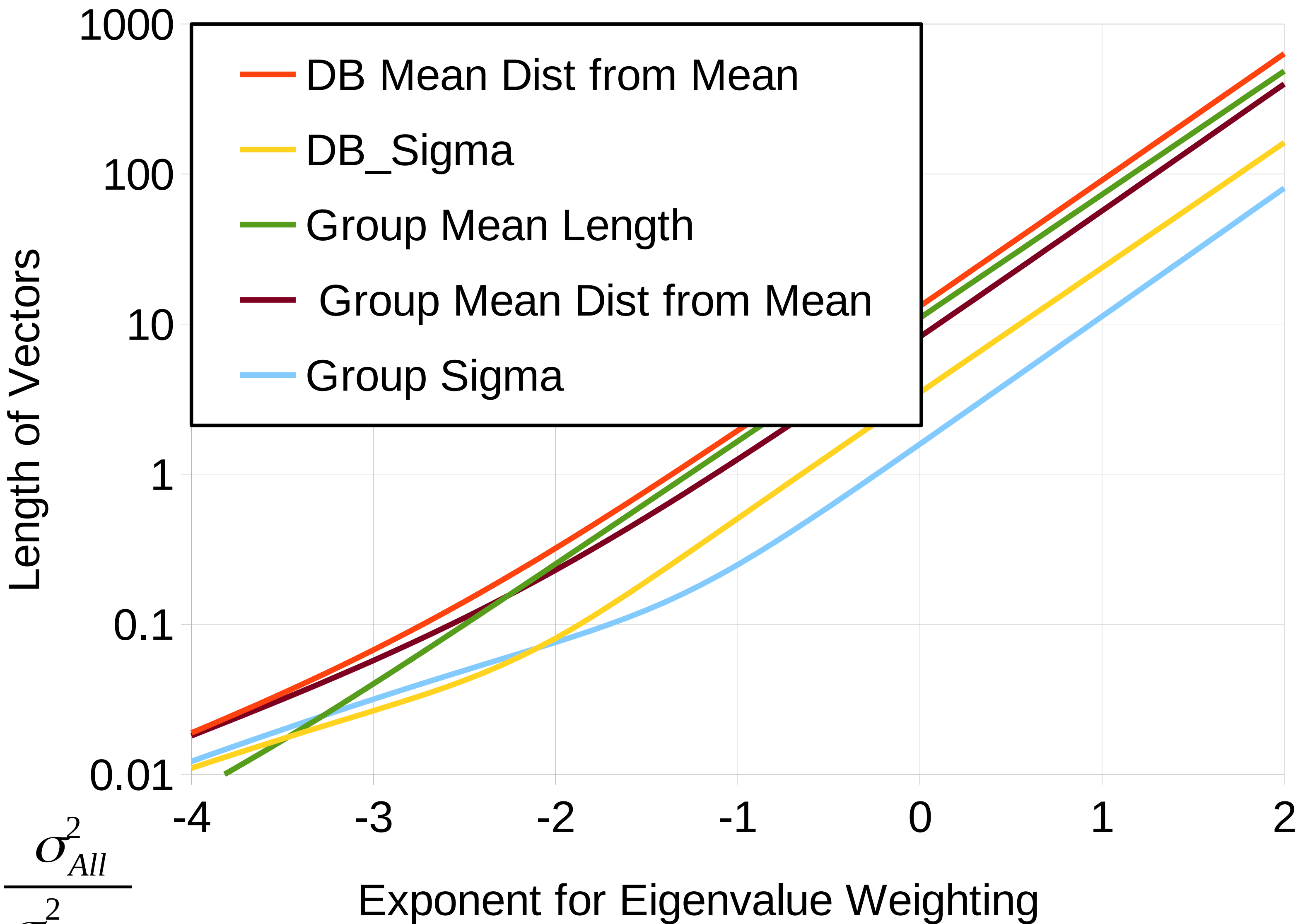
- MS distance from reference vector

$$\sigma^2 = \frac{1}{n-1} \cdot \sum_{1}^{n} \left(\left| \overrightarrow{cm_i} - \overrightarrow{cref} \right| - \bar{d} \right)^2$$

Statistics vs Weighting

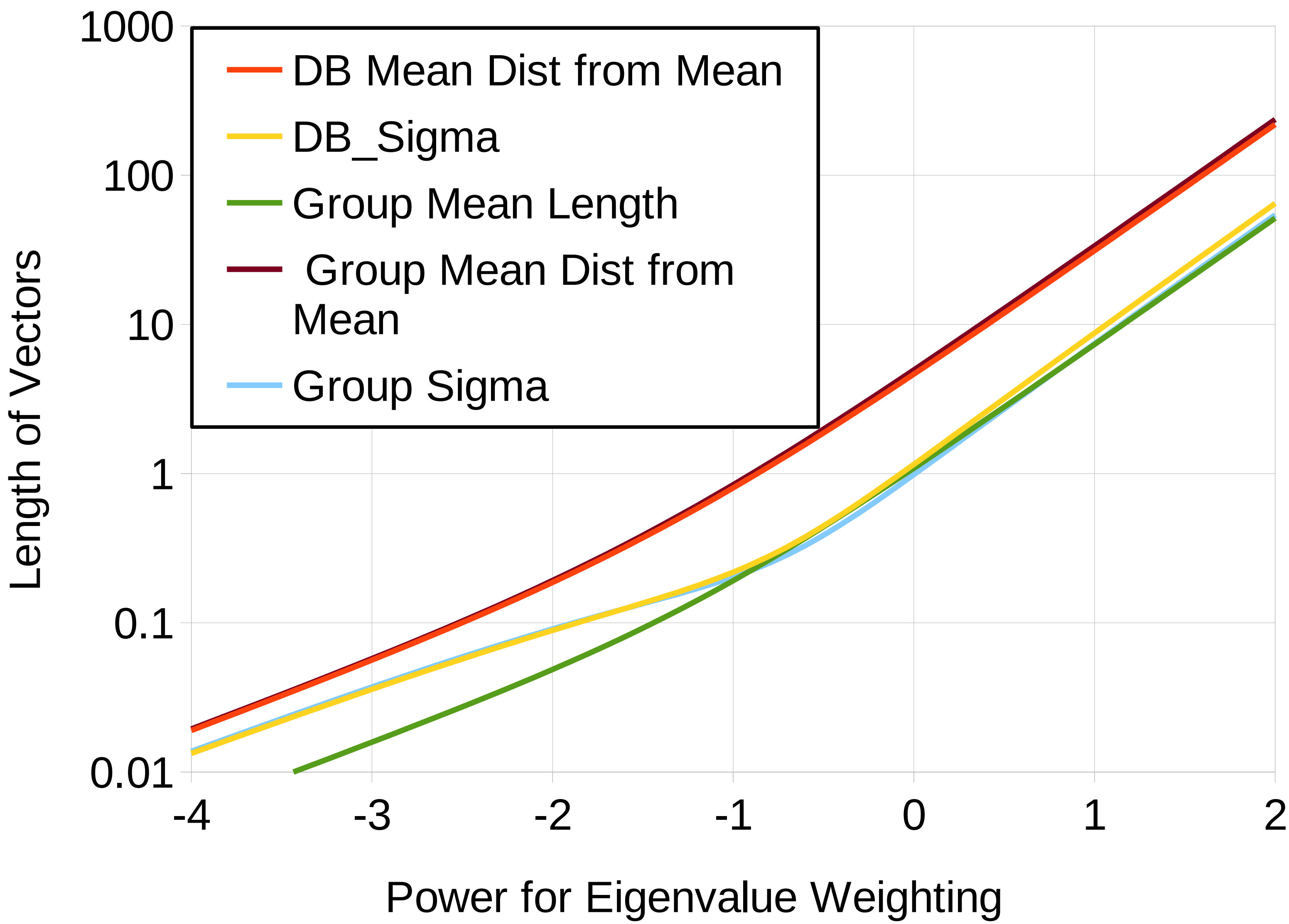


With Variance Weighting



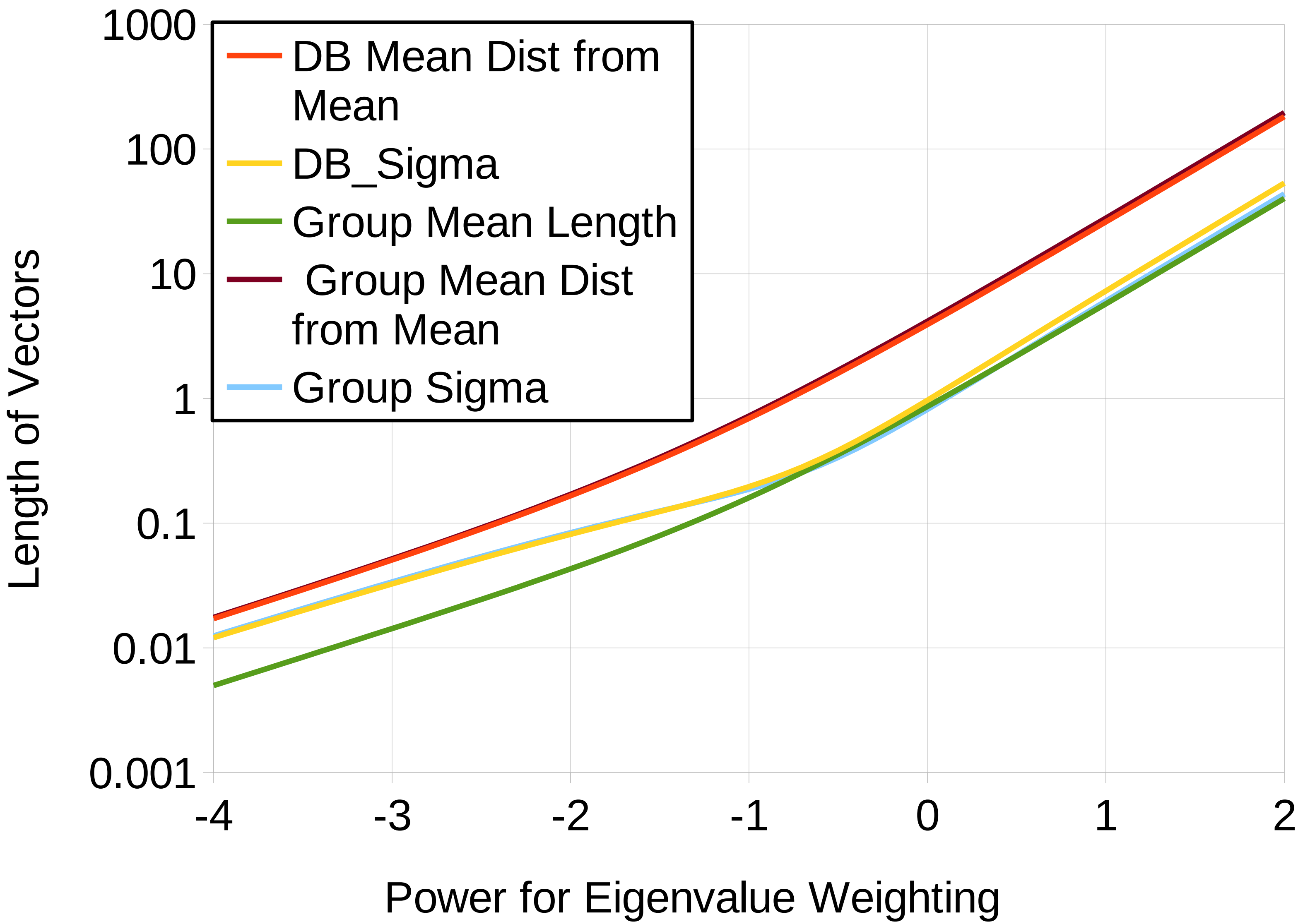
$$Ratio=\frac{\sigma_{All}^{2}}{\sigma_{Group}^{2}}$$

With Selected Eigenvectors



G > A

Variance Weighting & EV Selection



First Notes

- For the entire database, sum of mapped vectors is always zero
- When exponent is zero and eigenvector matrix is unweighted, all Costas arrays map to the same distance from the origin, variance is zero
- Mean distance to reference is greater than length of reference vector for any weighting
- Exponent of -1, as used with the left eigenvector matrix to reconstruct A, is mediocre

Unweighted Eigenvector Matrix

- Right eigenvector matrix is orthogonal

$$VR^T \cdot VR = VR \cdot VR^T = I$$

- Operating on vectors with orthogonal matrices
  - Preserves vector lengths
  - Preserves dot products
  - Preserves magnitudes of cross products
- Algorithm zeros first eigenvector
  - Corresponds to zero eigenvalue
  - Maps Costas array vector indexing from -(n-1)/2 to +(n-1)/2
  - All lengths are the same, variance is zero
- Not likely to be useful for Costas array selection without weighting

Why Sum of All in Database is Zero

- The Mapping
  - Mapping always zeros result of eigenvector that is all ones
  - This result carries only the information of whether indexing is from 0 to (n-1), or from 1 to n, and is of no value in selection for waveforms
- Full database will always contain complementary row Costas matrices
  - cm(i)=n-c(i)+1
  - Note that $\overrightarrow{cm}^T \cdot \overrightarrow{vr(j)} = -\overrightarrow{c}^T \cdot \overrightarrow{vr(j)}$

Eigenvalue Weighting of 0.5

- Base statistics
  - Reference vector length 8.3
  - Mean distance from reference 14.4
  - Mean square of distances from reference 2.54
- Compares with origin as reference, mean distance 17, mean square distance variation of 0.36
- Pick Costas arrays that map within 2.5 of reference vector, from 5.8 to 10.8

Selector Data File, Order 8 EP 0.5

8.279 Length of reference vector
14.44 Mean distance of mapped Costas array to reference vector
2.541 RMS distance of mapped Costas array to reference vector

0	0.5932	7.661	0.06717	0.329	-2.728	0.1936	1.385	Reference vector
0	-0.9083	-1.577	1.176	0	0	0	0	Row 1 of weighted IVR matrix
0	-0.6488	0.5257	-1.176	-1.535	1.005	0	0	Row 2 of weighted IVR matrix
0	-0.3893	0.5257	-0.7053	0.7676	-1.508	-1.355	0.6062	Row 3 of weighted IVR matrix
0	-0.1298	0.5257	-0.2351	0.7676	-0.5025	1.355	-1.818	Row 4 of weighted IVR matrix
0	0.1298	0.5257	0.2351	0.7676	0.5025	1.355	1.818	Row 5 of weighted IVR matrix
0	0.3893	0.5257	0.7053	0.7676	1.508	-1.355	-0.6062	Row 6 of weighted IVR matrix
0	0.6488	0.5257	1.176	-1.535	-1.005	0	0	Row 7 of weighted IVR matrix
0	0.9083	-1.577	-1.176	0	0	0	0	Row 8 of weighted IVR matrix

File name: Selector_N=8_EP=0.5000.csv

Eigenvalue Weighting of -4

- Base Statistics
  - Reference vector length 0.0057
  - Mean distance from reference 0.019
  - Mean square of distances from reference 0.014
- Compares with origin as reference, mean distance 0.019, mean square distance variation of 0.013
- Pick Costas arrays that map between 0.005 and 0.037 from reference vector

Eigenvector Weighting +4

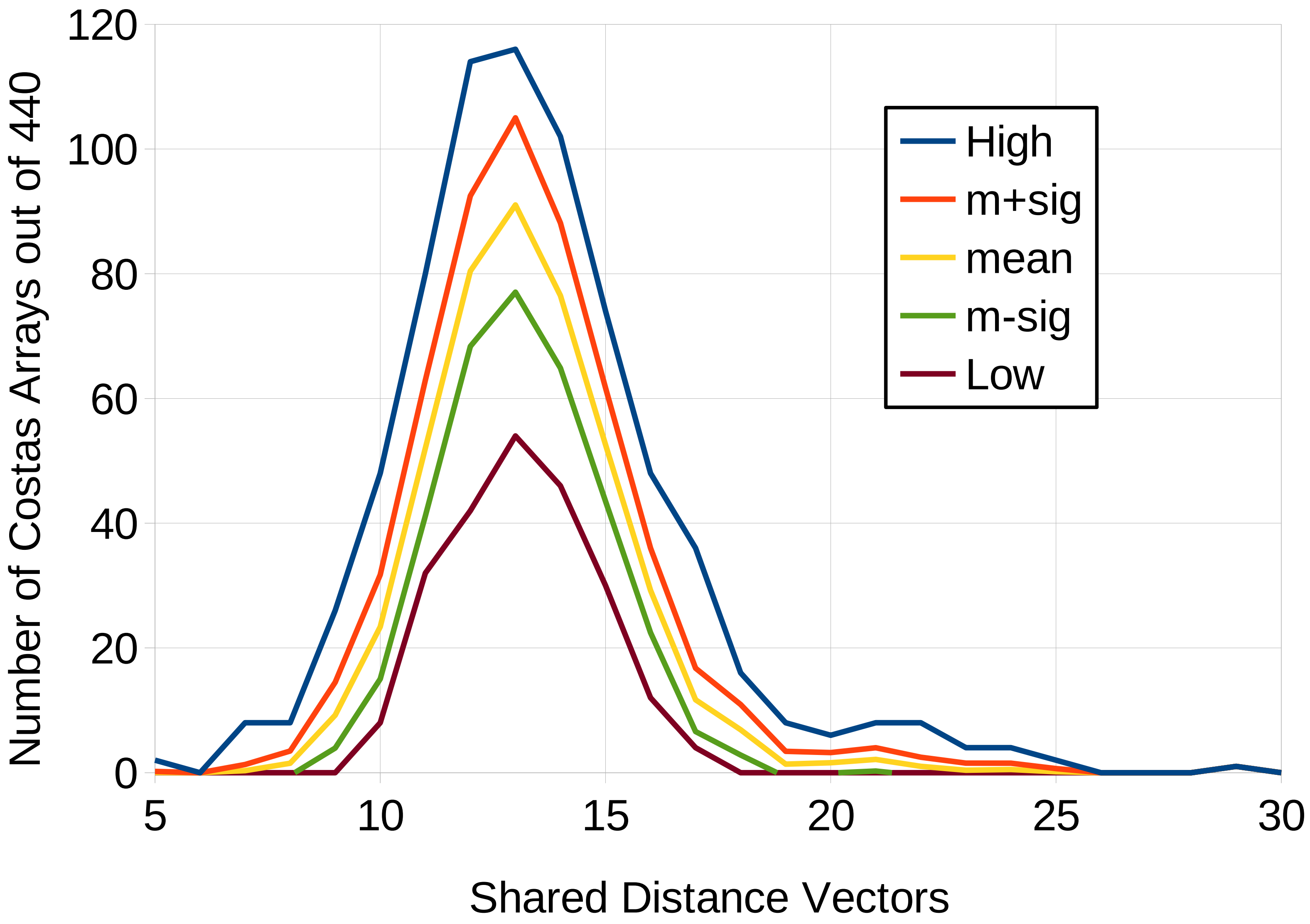
- Base statistics
  - Reference vector length 6,488
  - Mean distance from reference 13,650
  - Mean square variation of distances from reference 2510
- Compares with origin as reference, mean distance 15,830, mean square distance variation of 1,194
- Pick Costas arrays that map between 11,000 and 16,000 of reference vector

Which Weighting?

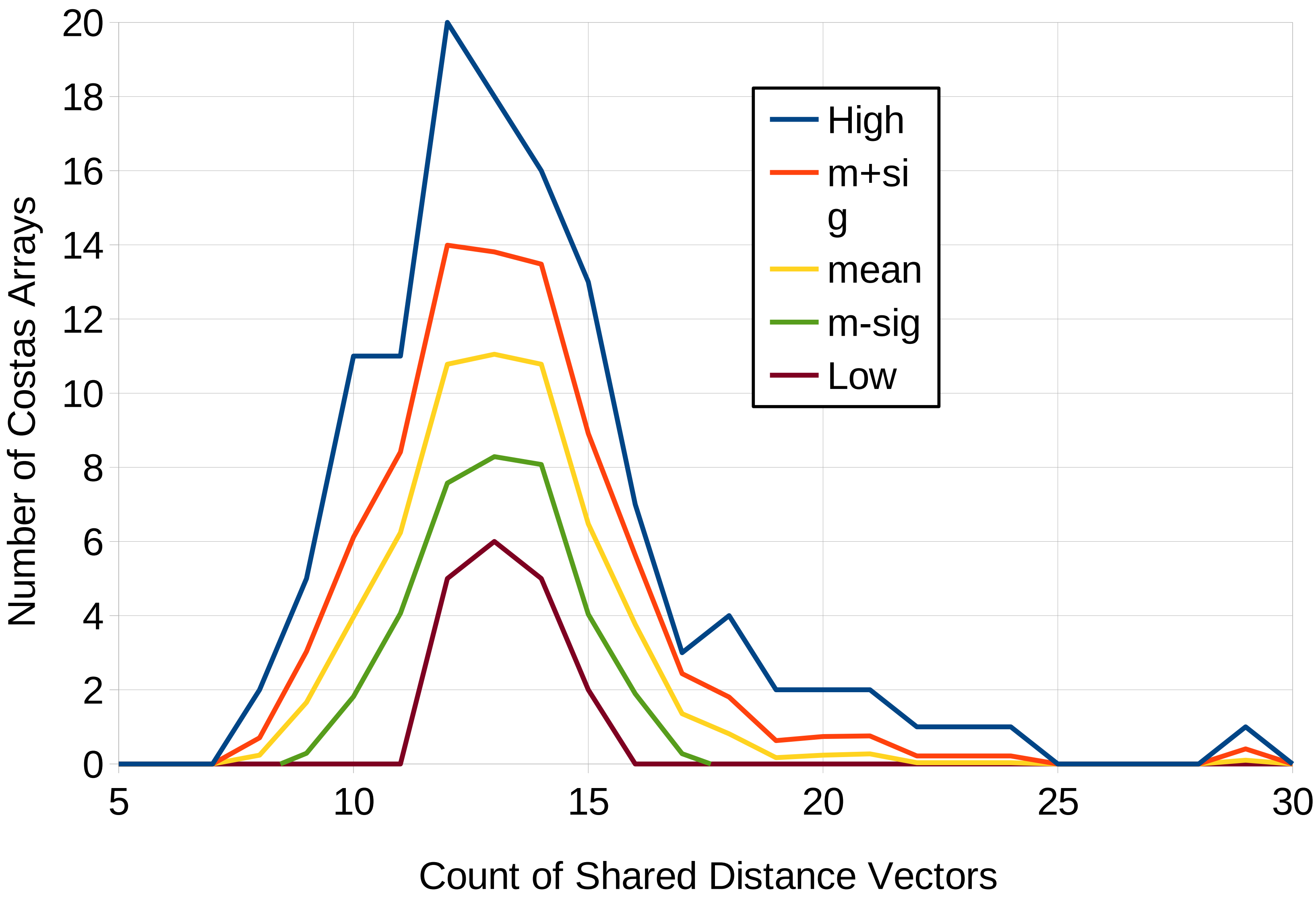
Using Three Selectors

- Use eigenvalue exponents -4, +0.5, and +4
- First, accept all Costas arrays in the database that have distances within one sigma of the reference vector
- Then, cull all Costas arrays in those results that do not meet all three criteria

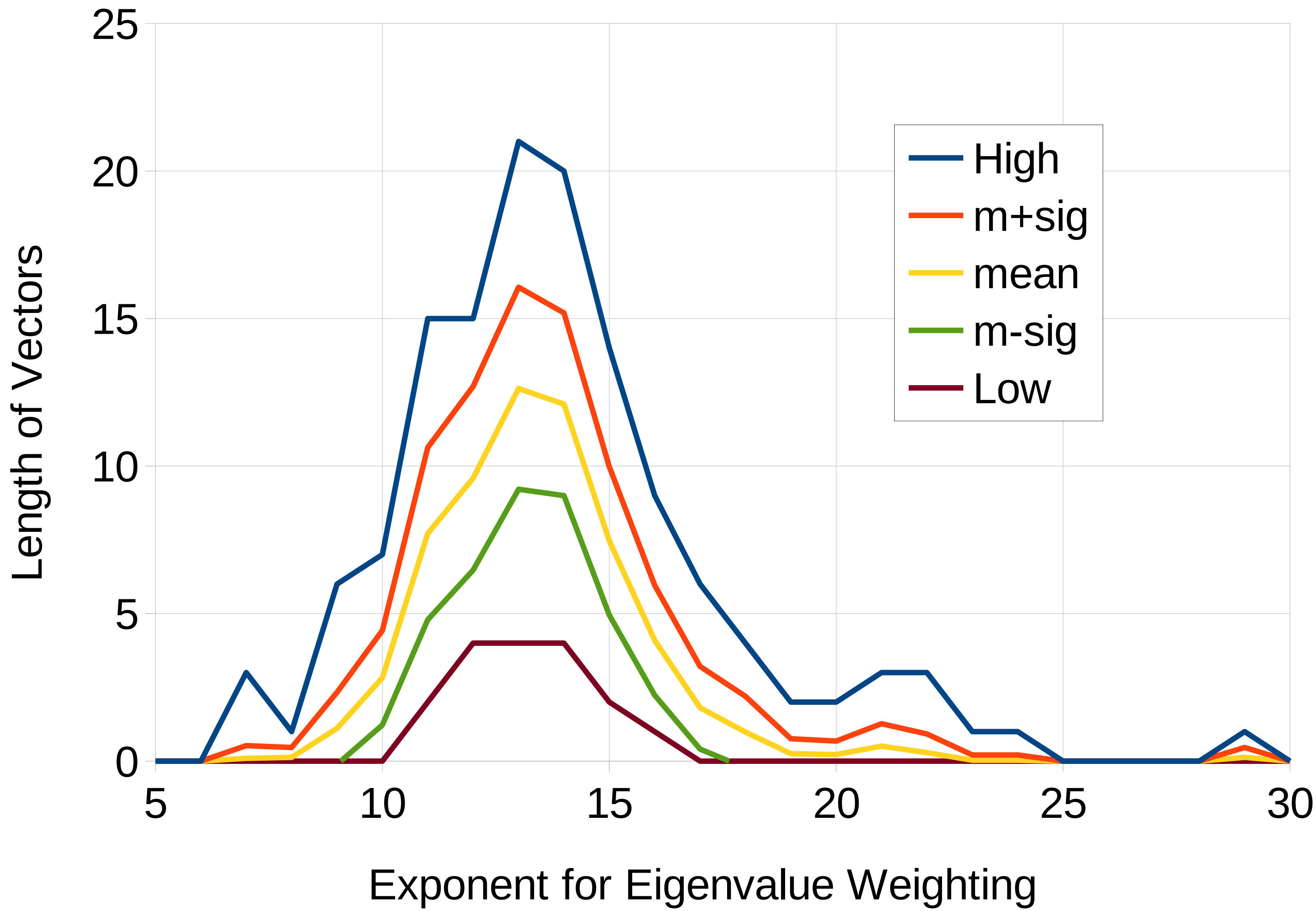
Order 8 Entire Database



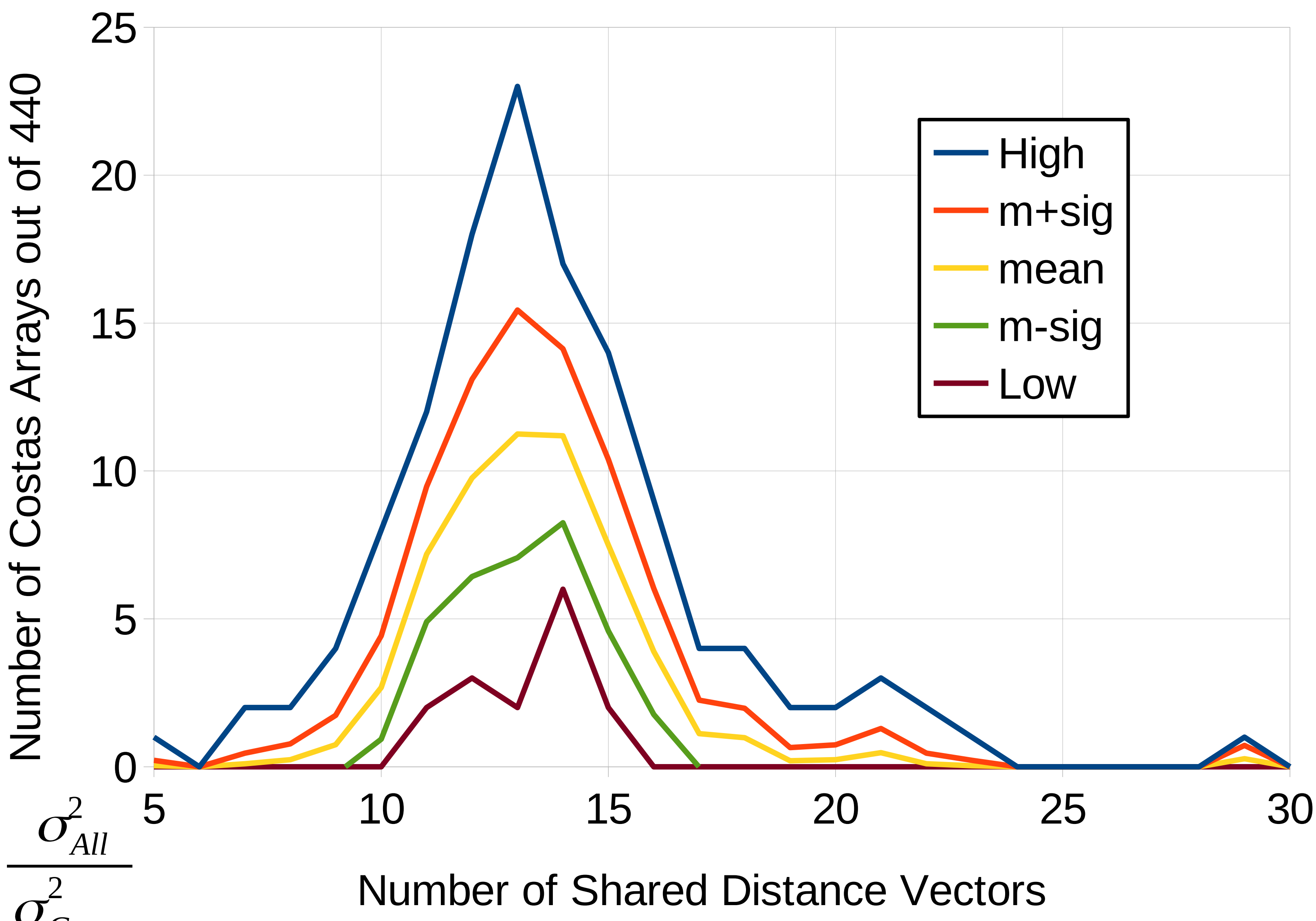
First Results



Eigenvalue Weighting Only

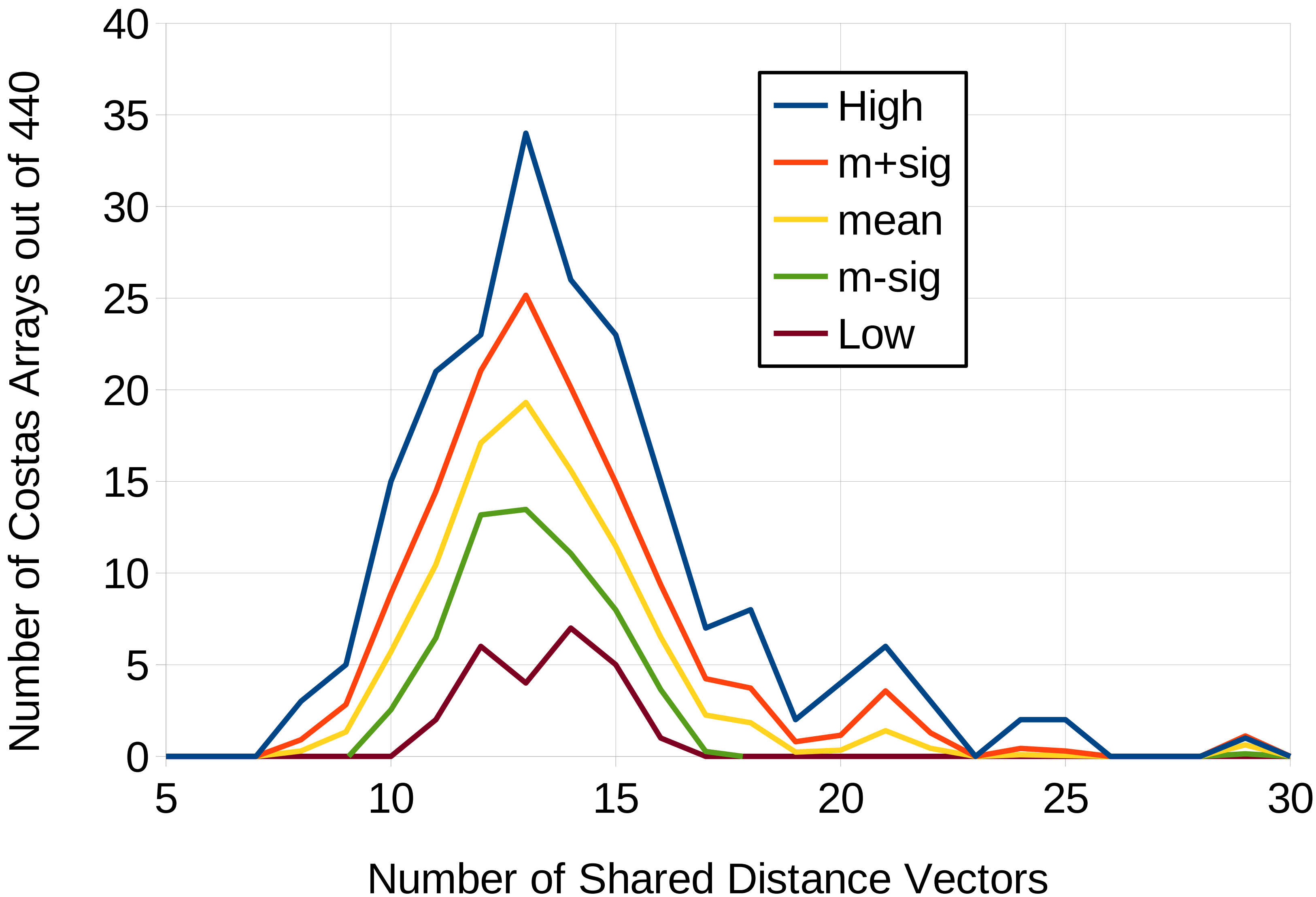


Variance Weighting



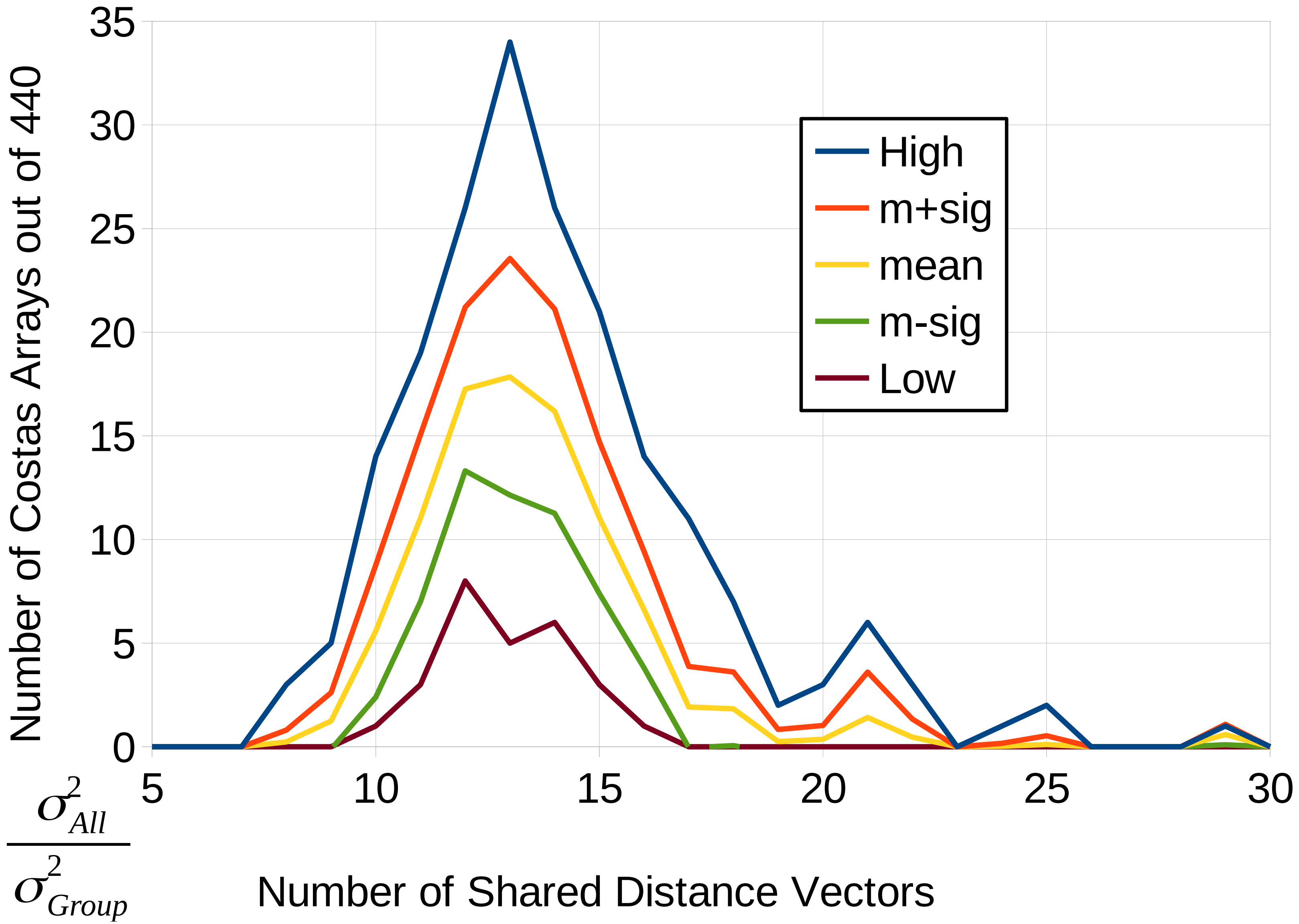
$$Ratio=\frac{\sigma_{All}^2}{\sigma_{Group}^2}$$

Selected Eigenvectors



G > A

Selected EV with Weighting



G > A

$$Ratio=\frac{\sigma_{All}^{2}}{\sigma_{Group}^{2}}$$

Artifacts of the Entire Cross-Matrix

- Vast majority share 13 or less distance vectors
- A few share more distance vectors
- Costas arrays 112 and 246 share all 28 distance vectors
- Measures to improve consistency
  - Simply pitch either number 112 or number 246
  - Reduce to less than two sigma width – do you need 59 Costas arrays in the group?

Other Selection Cost Functions

- Dot product with reference vector
- Projection orthogonal to reference vector

$$\left(I - \frac{\overrightarrow{vr} \cdot \overrightarrow{vr}^{T}}{\overrightarrow{vr}^{T} \cdot \overrightarrow{vr}} \right) \cdot \overrightarrow{cm}$$

- Assymetrical limits
  - Don’t allow too close to reference vector because Costas arrays will be too similar and will likely correlate
  - Perhaps allowing them farther won’t rapidly increase cross-correlations

Other Work (1 of 3)

- Extend search for low-correlation group
  - Existing method starts with designated Costas array
  - Search then selects others that meet cross-correlation criteria
  - Cross-correlations between Costas arrays in group used to cull group
  - Changing starting Costs array will produce a different group
- Use Jedwab/Wodliner dual/end-around distance vectors

Other Work (2 of 3)

- Use other weightings
  - Eigenvalue weightings is only a first example
  - Others such as selecting a weighted subset may perform better
- Use other cost functions
  - Current group definition cost function is shared distance vectors
  - Current selection cost function is Euclidean distance from a reference vector

Other Work (3 of 3)

- Use statistics of Costas array vectors in eigenspace
  - First cut takes mean plus and minus one sigma
  - We don't know what the distribution really looks like
  - We are taking the volume of a hollow hyperpolygon
  - Taking the volume of a hollow ellipsoid is far more exclusive, particularly for higher order
- We have just begun to scratch the surface here

Equations Containing Volumes

- Hyperpolygon

$$J_{HP} = \frac{\sum_{i=1}^{M} \left| v_1 - v_1 \right|}{\sqrt{\sigma_1^2 + \sigma_2^2}} \le Thresh_{HP}$$

- Hypercube

$$J_{HC} = Max_{1 \le i \le M} \frac{\left| v_1 - v_1 \right|}{\sqrt{\sigma_1^2 + \sigma_2^2}} \le Thresh_{HC}$$

- Hypersphere (Bhatacharya distance)

$$J_{HS} = \left(\vec{v}_1 - \vec{v}_2 \right)^T \cdot R^{-1} \cdot \left(\vec{v}_1 - \vec{v}_2 \right) \le Thresh_{HS}$$

Equations of Volumes

- Hyperpolygon

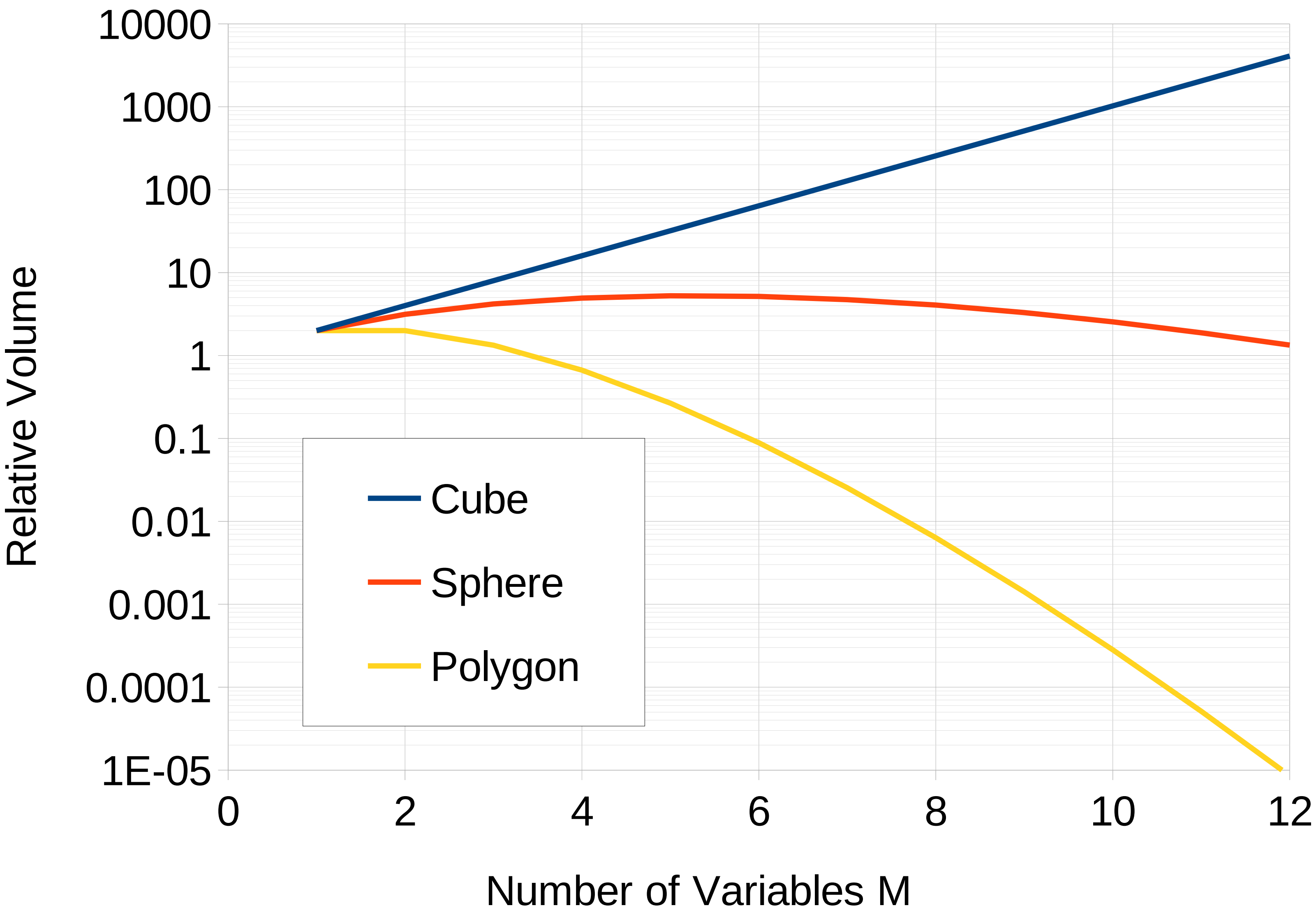
$$V_{HS} = \frac{2^M}{\Gamma(M+1)}$$

- Hypercube

$$V_{HC} = 2^M$$

- Hypersphere

$$V_{HS} = \frac{\pi^{M/2}}{\Gamma(M/2+1)}$$



10000
1000
100
10
1
0.1
0.01
0.001
0.0001
1E-05
Relative Volume
0
2
4
6
8
10
12
Number of Variables M
Cube
Sphere
Polygon

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Answer to Question

- “What language did you use in this work; was it MATLAB?
- Answer:
  - I used an object-oriented compiled language for this work
  - Database entries for the Costas arrays and SVD files are formatted for simple use with MATLAB or other languages
  - Database files are in CSV format, plain text with numbers separated by commas
  - On each line, all numerical data precedes ASCII comments
- The work presented TODAY could have been done using MATLAB