THE $\alpha - \beta - \eta - \theta$ TRACKER
WITH A RANDOM ACCELERATION PROCESS NOISE

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ABSTRACT: Time has proven that $\alpha - \beta$ filters provide a good and useful tracking methodology when used with sensors that measure position only. As new sensors are being integrated into platforms that measure position and Doppler velocity, a new tracker is required to properly process both position and velocity sensor measurements. This article addresses this need.

This article introduces an $\alpha-\beta-\eta-\theta$ filter, which processes both position and velocity sensor measurements. This new filter optimally tracks objects having a random acceleration maneuver (σ_A) model while being updated from a sensor having position (σ_p) and velocity(σ_v) measurement errors and T as the time between measurements. The key variables of this tracker, the gain and the covariance, are computed analytically. The filter gain is found to be a function of only the unitless position and velocity tracking indices Λ_p and Λ_v, defined as:

$$\Lambda_P \equiv \frac{\sigma_A T^2}{\sigma_P}$$

$$\Lambda_V \equiv \frac{\sigma_A T}{\sigma_V}.$$

Note that all the physics of the optimal gain is a function of only two ratios containing the only physical variables (σ_A, σ_P, σ_V, T) in the system, i.e., the tracking indices. Hence, two diverse systems with different measurements errors and different update times but with the same tracking indices will have the exact same optimal gains. The analytical form of the covariance is also computed. The covariance is shown to be equal to the gain matrix times the measurement error matrix.

The $\alpha - \beta - \eta - \theta$ filter computes the same results as the $\alpha - \beta$ filter [1] in the limit of position only updates or the velocity measurement error going to infinity.

Examples are computed that demonstrate that tracking errors are reduced if both position and velocity measurements are used in an $\alpha - \beta - \eta - \theta$ filter, rather than position only estimates in an $\alpha - \beta$ filter. This impacts the physical processes that use track kinematics.

GLOSSARY OF SYMBOLS

T	$\equiv$	The time between measurements
σ_A^2	$\equiv$	The random acceleration maneuver variance
σ_P^2	$\equiv$	The position measurement error variance
σ_V^2	$\equiv$	The velocity measurement error variance
Λ_p	$\equiv$	The unitless position tracking index
Λ_v	$\equiv$	The unitless velocity tracking index
K	$\equiv$	The gain of the tracking filter
$\alpha, \beta, \eta, \theta$	$\equiv$	The unitless variables in the gain matrix
Q	$\equiv$	The model process noise
$P_i(+)$	$\equiv$	The covariance after an update
a,b,c	$\equiv$	The components of the covariance after an update
$P_i(-)$	$\equiv$	The extrapolated covariance
F	$\equiv$	The transition matrix
H	$\equiv$	The measurement matrix
R	$\equiv$	The covariance of the measurement errors

INTRODUCTION

The $\alpha - \beta$ trackers have been used successfully for the past forty years [2] for tracking objects. The $\alpha - \beta$ tracker is obtained from the analytical solution of steady state Kalman tracker. The gains and covariance terms need not be computed recursively since they are a function of only the tracking index. This reduces the computation of the implementation. Normally, optimal tracking using an $\alpha - \beta$ filter or Kalman filter maintains a balance between the maneuver model error and the measurement error. Kalman filters in steady state have been used since 1976 [1] to design $\alpha - \beta$ trackers. The Kalman filter structure in steady state is the method used here to solve the $\alpha - \beta - \eta - \theta$ filter. The notation and the Kalman equations used in this analysis are taken from [3].

MATHEMATICAL MODEL

The gain of the $\alpha - \beta - \eta - \theta$ filter will have the following form, hence the name of the filter.

$$K = \begin{pmatrix} \alpha & (T\eta) \\ \left(\dfrac{\beta}{T}\right) & \theta \end{pmatrix} \qquad (1)$$

The model process noise is:

$$Q = \sigma_a^2 \begin{pmatrix} \dfrac{T^4}{4} & \dfrac{T^3}{2} \\ \dfrac{T^3}{2} & T^2 \end{pmatrix} \qquad (2)$$

The covariance after an update has the following form:

$$P_i(+) = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad (3)$$

The transition matrix is:

$$F = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \qquad (4)$$

The measurement matrix is:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (5)$$

The variance of the measurements assumes that no correlation exists between position and velocity measurements.

$$R_i = \begin{pmatrix} \sigma_p^2 & 0 \\ 0 & \sigma_v^2 \end{pmatrix} \qquad (6)$$

The extrapolated covariance is:

$$P_{i+1}(-) = FP_i(+)F^T + Q_i \qquad (7)$$

The result of substituting equations (2), (3), and (4) into (7) is:

$$P_{i+1}(-) = \begin{pmatrix} (a + 2bT + cT^2 + \sigma_a^2 T^4/4) & (b + cT + \sigma_a^2 T^3/2) \\ (b + cT + \sigma_a^2 T^3/2) & (c + \sigma_a^2 T^2) \end{pmatrix} \qquad (8)$$

The gain is computed as:

$$K_{i+1} = P_{i+1}(-)H_{i+1}^T[HP_{i+1}(-)H^T + R_{i+1}]^{-1} \qquad (9)$$

The covariance after an update is:

$$P_{i+1}(+) = [I - K_{i+1}H]P_{i+1}(-) \qquad (10)$$

Substituting equations (5) and (9) into (10) gives a new form for the covariance:

$$P_{i+1}(+) = P_{i+1}(-) - P_{i+1}(-)[P_{i+1}(-) + R_{i+1}]^{-1}P_{i+1}(-) \qquad (11)$$

Substituting equations (6) and (8) into (11) and simplifying leads to the following unique covariance terms:

$$P_{i+1}(+)_{(1,1)} = \frac{\sigma_p^2(-4b^2 + 4c\sigma_v^2 T^2 + c\sigma_a^2 T^4 + \sigma_a^2 \sigma_v^2 T^4 + 4a(c + \sigma_v^2 + \sigma_a^2 T^2) + 4b(2\sigma_v^2 T + \sigma_a^2 T^3))}{D_p} \qquad (12)$$

$$P_{i+1}(+)_{(1,2)} = P_{i+1}(+)_{(1,2)} = \frac{2\sigma_p^2 \sigma_v^2 (2b + 2cT + \sigma_a^2 T^3)}{D_p} \qquad (13)$$

$$P_{i+1}(+)_{(2,2)} = \frac{\sigma_v^2(-4b^2 + 4c\sigma_p^2 + 4\sigma_a^2\sigma_p^2T^2 + 4b\sigma_a^2T^3 + c\sigma_a^2T^4 + 4a(c + \sigma_a^2T^2))}{D_P} \qquad (14)$$

with

$$D_P \equiv -4b^2 + 4c\sigma_p^2 + 4\sigma_p^2\sigma_v^2 + 4\sigma_a^2\sigma_p^2T^2 + cT^2(4\sigma_v^2 + \sigma_a^2T^2) + \sigma_a^2\sigma_v^2T^4 + 4a(c + \sigma_v^2 + \sigma_a^2T^2) + 4b(2\sigma_v^2T + \sigma_a^2T^3)$$

Substituting $\sigma_P \equiv \dfrac{\sigma_A T^2}{\Lambda_P}$ and $\sigma_V \equiv \dfrac{\sigma_A T}{\Lambda_V}$ into the above equations and setting $P_i(+) = P_{i+1}(+)$

leads to the solutions of the three steady state covariance components a, b, and c of equation 3,:

$$a = P(+)_{(1,1)} = \left(\frac{-\Lambda_p^2\left(16 + 2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4}\right) + \Lambda_p\left(8 + \Lambda_p\sqrt{\Lambda_v^2 + 4}\right)\sqrt{\Lambda_p^2 + 4(\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})}}{8\Lambda_p^2(2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})} \right)\sigma_a^2T^4 \qquad (15)$$

$$b = P(+)_{(1,2)} = P(+)_{(2,1)} = \left(\frac{8 + 2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4} - \sqrt{\Lambda_v^2 + 4}\sqrt{\Lambda_p^2 + 4(\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})}}{4(2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})} \right)\sigma_a^2T^3 \qquad (16)$$

$$c = P(+)_{(2,2)} = \left(\frac{\sqrt{\Lambda_v^2 + 4}\sqrt{\Lambda_p^2 + 4(\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})}}{2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4}} - 1 \right)\frac{\sigma_a^2T^2}{2} \qquad (17)$$

The steady state extrapolated covariance can be computed using equation 7.

$$P(-) = FP(+)F^T + Q \qquad (18)$$

Simplifying the results, the covariance components are:

$$P(-)_{(1,1)} = \left(\frac{\Lambda_p^2\left(16 + 2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4}\right) + \Lambda_p\left(8 + \Lambda_p\sqrt{\Lambda_v^2 + 4}\right)\sqrt{\Lambda_p^2 + 4(\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})}}{8\Lambda_p^2(2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})} \right)\sigma_a^2T^4 \qquad (19)$$

$$P(-)_{(1,2)} = P(-)_{(2,1)} = \left(\frac{8 + 2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4} + \sqrt{\Lambda_v^2 + 4}\sqrt{\Lambda_p^2 + 4(\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})}}{4(2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})} \right)\sigma_a^2T^3 \qquad (20)$$

$$P(-)_{(2,2)} = \left(\frac{\sqrt{\Lambda_v^2 + 4}\sqrt{\Lambda_p^2 + 4(\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})}}{2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4}} + 1 \right)\frac{\sigma_a^2T^2}{2} \qquad (21)$$

Rewriting the variance of the measurements of equation (6) as:

$$R = \begin{pmatrix} \dfrac{\sigma_A^2T^4}{\Lambda_P^2} & 0 \\ 0 & \dfrac{\sigma_A^2T^2}{\Lambda_V^2} \end{pmatrix}, \qquad (22)$$

and substituting to compute the gain as:

$$K = P(-)H[HP(-)H + R]^{-1} \qquad (23)$$

The components of the gain after simplification are:

$$\alpha = K_{(1,1)} = \left(\frac{-\Lambda_p^2\left(16 + 2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4}\right) + \Lambda_p\left(8 + \Lambda_p\sqrt{\Lambda_v^2 + 4}\right)\sqrt{\Lambda_p^2 + 4(\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})}}{8(2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})} \right) \qquad (24)$$

$$\eta T = K_{(1,2)} = \left(\frac{8 + 2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4} - \sqrt{\Lambda_v^2 + 4}\sqrt{\Lambda_p^2 + 4(\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})}}{4(2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})} \right)\Lambda_v^2T \qquad (25)$$

$$\frac{\beta}{T} = K_{(2,1)} = \left(\frac{8 + 2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4} - \sqrt{\Lambda_v^2 + 4}\sqrt{\Lambda_p^2 + 4(\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})}}{4(2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})T} \right)\Lambda_p^2 \qquad (26)$$

$$\theta = K_{(2,2)} = \left(\frac{\sqrt{\Lambda_v^2 + 4}\sqrt{\Lambda_p^2 + 4(\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4})}}{2\Lambda_v^2 + \Lambda_p\sqrt{\Lambda_v^2 + 4}} - 1 \right)\frac{\Lambda_v^2}{2} \qquad (27)$$

An interesting relationship between the off diagonal gain components is:

$$\frac{K_{(2,1)}}{K_{(1,2)}} = \frac{\Lambda_p^2}{\Lambda_v^2T^2} = \frac{\sigma_v^2}{\sigma_p^2} \qquad (28)$$

Or equivalently between β and η :

$$\beta = \left(\frac{\sigma_v^2T^2}{\sigma_p^2} \right)\eta = \left(\frac{\Lambda_p^2}{\Lambda_v^2} \right)\eta \qquad (29)$$

The covariance can be shown to be equal to the gain times the measurement error matrix:

$$P(+)_{(1,1)} = K_{(1,1)}\sigma_p^2 \qquad (30)$$

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$$P(+)_{(1,2)} = K_{(1,2)}\sigma_v^2 \qquad (31)$$

$$P(+)_{(2,1)} = K_{(2,1)}\sigma_p^2 \qquad (32)$$

$$P(+)_{(2,2)} = K_{(2,2)}\sigma_v^2 \qquad (33)$$

or equivalently:

$$P(+) = K \cdot R \qquad (34)$$

This form can also be used to compute the gain.

$$K = P(+) \cdot \begin{pmatrix} 1/\sigma_p^2 & 0 \\ 0 & 1/\sigma_v^2 \end{pmatrix} \qquad (35)$$

This is consistent with the following equation from [3]:

$$K_i = P_i(+) \cdot H_i^T \cdot R_i^{-1} \qquad (36)$$

A plot of the gains as a function of the positional and velocity tracking indices Λ_p and index Λ_v is shown in the following plots from figure 1 to figure 4.
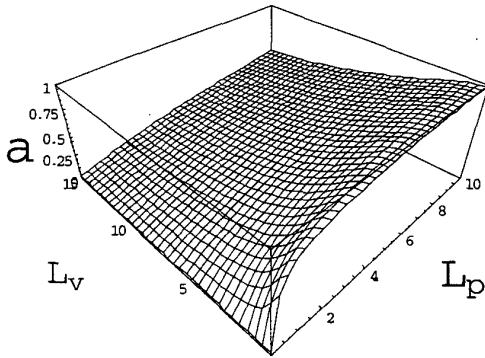
Figure 1 is a plot of the gain component α (equation 24) as a function of the positional tracking index Λ_p from 0 to 10 and of the velocity tracking index Λ_v from 0 to 15
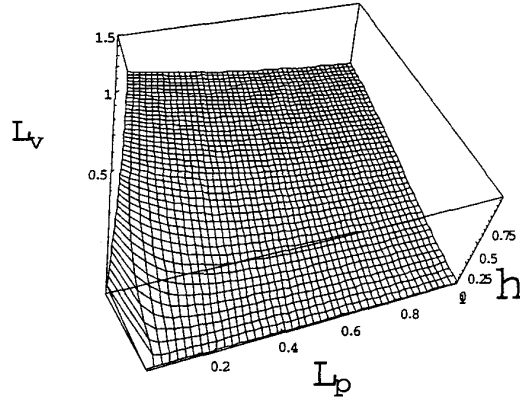

Figure 2 is a plot of the gain component η (equation 25) as a function of the positional tracking index Λ_p from 0 to 1 and of the velocity tracking index Λ_v from 0 to 1.5
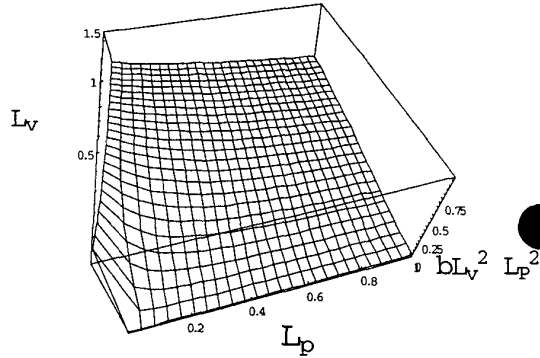

Figure 2a is a plot of the function $\Lambda_v^2 \beta / \Lambda_p^2$ as a function of the positional tracking index Λ_p from 0 to 1 and of the velocity tracking index Λ_v from 0 to 1.5
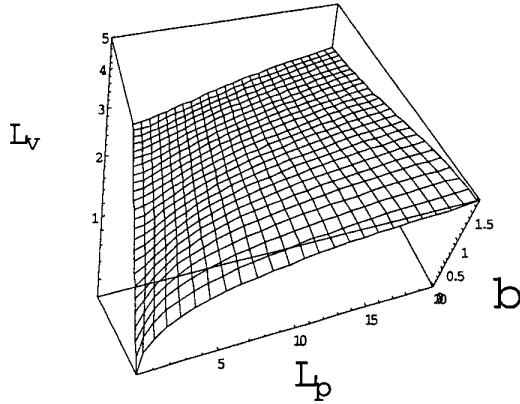
Figure 3 is a plot of the gain component β (equation 26) as a function of the positional tracking index Λ_p from 0 to 20 and of the velocity tracking index Λ_v from 0 to 5
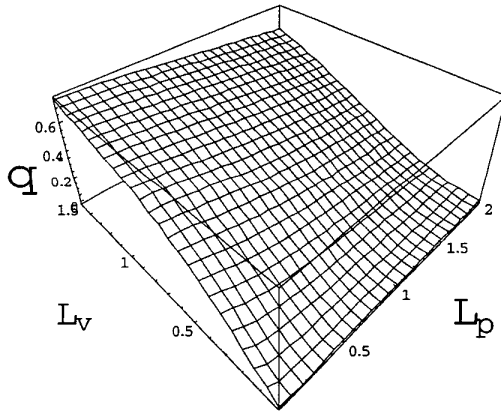

Figure 4 is a plot of the gain component θ (equation 27) as a function of the positional tracking index Λ_p from 0 to 2 and of the velocity tracking index Λ_v from 0 to 1.5

In the Limit of $\alpha - \beta - \eta - \theta$ with Position Measurements Only

The tracking performance of the $\alpha - \beta - \eta - \theta$ in the limit of the velocity measurement error going to

infinity, is equivalent to an $\alpha - \beta$ tracking filter with only position measurements. This tracker was analyzed in 1976 [1] where the $\alpha - \beta$ filter has a random acceleration maneuver σ_A model while being updated from a sensor having raw positional measurements estimates with an error of σ_P. Therefore, in the limit of the velocity measurement error going to infinity is equivalent to the velocity tracking index going to zero.

$$\Lambda_V \equiv \frac{\sigma_A T}{\sigma_V} \qquad (37)$$

$$\Lambda_V = 0$$

In the limit of only positional updates, the velocity tracking index goes to zero and the non zero gains are:

$$\alpha_0 = \left(\frac{-\Lambda_p(8 + \Lambda_p) + (4 + \Lambda_p)\sqrt{\Lambda_p(8 + \Lambda_p)}}{8} \right) \qquad (38)$$

$$\beta_0 = \left(\frac{4 + \Lambda_p - \sqrt{\Lambda_p(8 + \Lambda_p)}}{4} \right) \Lambda_p \qquad (39)$$

The above gains solve the [1] constraint equations.

$$\frac{\beta_0^2}{1 - \alpha_0} = \Lambda_p^2 \qquad (40)$$

and

$$\beta_0^2 - 4(2 - \alpha_0)\beta_0 + 4\alpha_0^2 = 0, \qquad (41)$$

or equivalently

$$\beta_0 = 2(2 - \alpha_0) - 4\sqrt{1 - \alpha_0} \qquad (42)$$

Therefore in the limit of only positional measurements the $\alpha - \beta - \eta - \theta$ tracker gives the correct $\alpha - \beta$ tracker results.

EXAMPLE

In this section an example will be calculated comparing a tracking filter that updates using only position measurements to one that updates using both position and velocity measurements. The sensor position measurement error is the same for both trackers. The requirement on the track maneuver is the same as the value for the random acceleration maneuver (σ_A) model.

Case 1. The tracking filter updates with position and velocity measurements.

$$T = 1\,sec$$
$$\sigma_A = 10\ ft\ /\ sec^2$$
$$\sigma_p = 500\ ft$$
$$\sigma_v = 5\ ft\ /\ sec$$

The gain is computed as:

$$K = \begin{pmatrix} 0.00992129 & 0.578782\ sec \\ 0.0000578782\ /\ sec & 0.828392 \end{pmatrix}$$

$$P(+) = \begin{pmatrix} 2480\ ft^2 & 14.5\ ft^2\ /\ sec \\ 14.5\ ft^2\ /\ sec & 20.7\ ft^2\ /\ sec^2 \end{pmatrix}$$

Using the square root of the covariance as a measure of the performance for the tracker, the one sigma position error is 49.8 ft and the velocity error is 4.6 ft/sec.

Case 1A. The tracking filter updates with position measurements only.

$$T = 1\,sec$$
$$\sigma_A = 10\ ft\ /\ sec^2$$
$$\sigma_p = 500\ ft$$

The gain is computed as:

$$K = \begin{pmatrix} 0.181201 \\ 0.0180975\ /\ sec \end{pmatrix}$$

$$P(+) = \begin{pmatrix} 45300\ ft^2 & 4524\ ft^2\ /\ sec \\ 4524\ ft^2\ /\ sec & 951\ ft^2\ /\ sec^2 \end{pmatrix}$$

Using the square root of the covariance as a measure of the performance for the tracker, the one sigma position error is 212.8 ft and the velocity error is 30.8 ft/sec.

A comparison of this example shows that using the velocity measurements **reduces** the position error estimates of the tracker by a factor of four and it also **reduces** the velocity error estimates of the tracker by a factor of six.

Case 2. The tracking filter updates with position and velocity measurements.

$$T = 1\,sec$$
$$\sigma_A = 10\ ft\ /\ sec^2$$
$$\sigma_p = 50\ ft$$
$$\sigma_v = 5\ ft\ /\ sec$$

The gain is computed as:

$$K = \begin{pmatrix} 0.0926147 & 0.521288\ sec \\ 0.00521288\ /\ sec & 0.825341 \end{pmatrix}$$

$$P(+) = \begin{pmatrix} 231.5\ ft^2 & 13.0\ ft^2\ /\ sec \\ 13.0\ ft^2\ /\ sec & 20.6\ ft^2\ /\ sec^2 \end{pmatrix}$$

Using the square root of the covariance as a measure of the performance for the tracker, the one sigma position error is 15.2 ft and the velocity error is 4.5 ft/sec.

Case 2A. The tracking filter updates with position measurements only.

$$T = 1\,sec$$
$$\sigma_A = 10\ ft\ /\ sec^2$$
$$\sigma_p = 50\ ft$$

The gain is computed as:

$$K = \begin{pmatrix} 0.467328 \\ 0.145969\ /\ sec \end{pmatrix}$$

$$P(+) = \begin{pmatrix} 1168\ ft^2 & 365\ ft^2\ /\ sec \\ 365\ ft^2\ /\ sec & 270\ ft^2\ /\ sec^2 \end{pmatrix}$$

Using the square root of the covariance as a measure of the performance for the tracker, the one sigma position error is 34 ft and the velocity error is 16 ft/sec.

A comparison of this example shows that using the velocity measurements **reduces** the position error estimates of the tracker by a factor of two and it also **reduces** the velocity error estimates of the tracker by a factor of three.

CONCLUSION

This article introduces and solves the $\alpha - \beta - \eta - \theta$ filter. This filter is used to optimally track objects having a random acceleration maneuver (σ_A) model while being updated from sensors having position and velocity measurements estimates with respective errors of (σ_P) and (σ_V). The filter gains are a function of only the unitless position and velocity tracking indices

$$\Lambda_P \equiv \frac{\sigma_A T^2}{\sigma_P}$$

$$\Lambda_V \equiv \frac{\sigma_A T}{\sigma_V}.$$

The closed form solution of the covariance has been computed. The covariance is shown to be equal to the gain times the measurement error matrix, as expected from Kalman filter theory.

The $\alpha - \beta - \eta - \theta$ filter correctly computes the same results as the $\alpha - \beta$ filter [1] in the limit of only position updates or the velocity measurement error going to infinity.

An example was computed that demonstrates that tracking errors are reduced if both position and velocity measurements are used in an $\alpha - \beta - \eta - \theta$ filter, rather than position only estimates in an $\alpha - \beta$ filter. This impacts the physical processes.

REFERENCES

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