What a long strange trip it has been: My twenty-five year affair with the alpha-beta filter.

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Some careers are defined by ones major in college, other careers are defined by the first problem one is given to solve independently (the problem the boss wants to know the answer to and it is your job to find.). My first problem led to my meeting and interacting with Fellows of the IEEE, Nobel Prize winners, near Nobel prize winners, people with Erdos numbers of one, and my wife for that matter. This problem was first posed to me twenty five years ago when I first started working at what is now the Naval Surface Warfare Center in Dahlgren, Virginia. I started working at Dahlgren after graduating from the University of Mississippi with three degrees, two BS (Physics and Mathematics) and a Masters degree in physics. In the time I was at Mississippi I never managed to specialize in anything. Besides the standard graduate curriculum in physics (mechanics, quantum mechanics, electromagnetism) I also managed to supplement my undergraduate mathematics training with course work useful to physics. I also indulged in such esoteric mathematics as continued fractions, number theory, and group theory in physics , it seems less so looking backwards, but seemed strange to advisors at the time. While an undergraduate, I managed to minor in history and philosophy as well (including a flirtation with chemistry initially that did not go well). My father, who was on the staff of the university, later told me that I came close to setting a record for the number of undergraduate courses I had taken.

After taking a final graduate course the first summer semester and finishing my degree in July 1980, I spent the second semester of summer teaching at a small liberal arts college in Jackson Mississippi teaching the second semester of a premed physics course. That experience convinced me that I had no desire to be a teacher, so I turned down an offer to join the physics department at a nearby junior college. I had a job offer from a government organization in Texas and one from the Navy in Dahlgren Virginia in a group that worked on radar. I was born in Virginia, but had lived there only briefly before moving to Germany (Army brat). I decided on Dahlgren because I thought that working in radar would scratch both my mathematics and physics itch.

The canonical picture for the way a sensor works is similar to the canonical Shannon's diagram of a communication system. The transmitter is replaced in the diagram with a sensor that acts as the broadcast source. The information source is moved to the right of the noise injection input so the information source interacts with transmitted signal. The information is then captured by the receiver (electromagnetic, acoustical, or some other type of wave phenomena) and is passed on the destination which is then processed by the user of the transmitted signal. Most practical sensors have to track multiple objects by scanning a three (two) dimensional volume, so they divide the volume up into sectors and scan for objects sector by sector. Trying to determine whether a target is present in a sector (detection) requires expenditure of sensor resources in both time and energy. Dwell time is the amount of time a sensor expends to search a particular sector. At the same time, broadcast power times the time spent is the dwell energy expended by the sensor. Once a target has been detected repeatedly, there is then an advantage to keeping a record of what it detects using a recursive record maintained by a computer in the loop that points the sensor (tracking). Attempting to understand the properties of this recursive record was one of the first problems I encountered during the professional development stage of my career. There are many lessons I learned from this type of problem, which still holds some interest to me and may interest the reader.

The simplest type of tracking filter widely used in radar systems is the $\alpha - \beta$ filter. The filter has equations that consist of two parts: prediction equations, which are given by (note the implicit assumption that the target is moving at a constant velocity)

$$x_p(k) = x_s(k-1) + v_s(k-1)T \tag{1}$$

$$v_p(k) = v_s(k-1) \tag{2}$$

and smoothing equations, which are given by

$$x_s(k) = x_p(k) + \alpha(x_m(k) - x_p(k)) \tag{3}$$

$$v_s(k) = v_p(k) + \frac{\beta}{T}(x_m(k) - x_p(k)) \tag{4}$$

where

- $x_s(k)$ = smoothed position at the k-th interval
- $x_p(k)$ = predicted position at the k-th interval

- $x_m(k)$ = measured position at the k-th interval
- $v_s(k)$ = smoothed velocity at the k-th interval
- $v_p(k)$ = predicted velocity at the k-th interval
- T = radar update interval
- α, β = filter weighing coefficients

The question of the selection of filter coefficient values and the relationship between the coefficients used in tracking filters which are used to derive pointing commands for a radar dates back at least as far as work by Sklansky [20]. Sklansky proposed measures of performance that include *stability*, *transient response*, *noise* and *maneuver error* as a function of the dynamic parameters α and β. Noise reduction means the response of the filter to a stochastic input with respect to the measurement process. These responses are called noise reduction ratios, which represent the mean square error about the average measurement. The position noise reduction ratio is

$$P_x(0) = \frac{2\alpha^2 - \beta(3\alpha - 2)}{\alpha(4 - 2\alpha - \beta)}, \tag{5}$$

while the velocity noise reduction ratio is

$$P_v(0) = \frac{2\beta^2}{T^2\alpha(4 - 2\alpha - \beta)}. \tag{6}$$

For values of α and β that are used in tracking, these ratios are less than one, so uncertainty about the mean is effectively reduced by the use of the filter on measurements, which is why they are used. Similarly one can also determine the response of the filter to unmodeled or stochastic behavior of the model of the target behavior, which are the transient noise reduction ratios; the transient noise reduction ratio for the position

$$T_x(0) = \frac{(2 - \alpha)(1 - \alpha)^2 T^2}{\beta\alpha(4 - 2\alpha - \beta)}, \tag{7}$$

while the transient noise reduction ratio for the velocity is

$$T_v(0) = \frac{\alpha^2(2 - \alpha) + 2\beta(1 - \alpha)}{\beta\alpha(4 - 2\alpha - \beta)}. \tag{8}$$

The thing to note about this is that the transient behavior is the opposite of noise behavior, small α produces more noise reduction while increasing transient response, large α produces less noise reduction, while improving transient response. Assuming that a filter design should deal effectively with both of these criteria, minimizing mean square error associated with sensor measurement and minimize the response of the filter to unmodeled target behavior, one is left with a dilemma, these goals are contradictory.

All of the early work to analyze filter behavior was based on a frequency domain or Z-transform analysis (discrete version of the Laplace transform). Benedict-Bordner [4] proposed a relationship between α and β based on a pole-matching technique that combined transient performance and noise reduction capability which was later called the Benedict-Bordner relationship

$$\beta = \frac{\alpha^2}{2-\alpha}. \tag{9}$$

By proposing a relationship between α and β, $\beta = \beta(\alpha)$, a solution is obtained that represents a solution to the tracking dilemma. To define a relationship $\beta = \beta(\alpha)$ amounts to a choice that represents a compromise between contradictory goals. My boss (then Mr. now Dr. Terry Foreman) showed me a book on radar systems that had the tracking equations that explained why they are being used by radar systems and a copy of the paper by Benedict. Some engineers at RCA, later General Electric, and currently Lockheed-Martin had proposed replacing it by the relationship (now called the Kalata relationship named for Dr. Paul Kalata now of Drexel University) first publicly proposed in [12])

$$\beta = 2\,(2-\alpha) - 4\sqrt{1-\alpha} \tag{10}$$

My problem was to figure out where this equation came from and then determine whether or not it was a good idea. The problem has occupied my attention on and off for the last twenty five years. My first resolution as to where the Kalata relationship came from trying to understand the Kalman filter, which led to an explanation for this relationship. Analysis performed by Simpson [19], Neal, and Benedict [16] extended the analysis by Benedict to the $\alpha - \beta - \gamma$ filter, which is the generalization to a tracker designed to track accelerating threats. By this time, the Kalman filter was becoming well known in the radar community. Thereafter, the tendency is to discuss the $\alpha - \beta$ and $\alpha - \beta - \gamma$ filters as steady state solutions to the Kalman filter. (Note, I find it useful to consider the $\alpha - \beta$ filter independently of the Kalman filter, and introduce the Kalman filter as a generalization of the $\alpha - \beta$ filter, so I will not connect the two in this note.)

I was able to exploit this work to arrive at the Kalata relationship. This led to me visiting RCA and meeting the person responsible for this change to the design of the radar tracking equations. (Dr. Frank Reifler, a mathematician whose thesis was in algebraic topology, is responsible for much of the design of the trackers used at that time in the surface Navy premier destroyer fleet, the Aegis system, as well as many of the other important estimation algorithms. He has twice won the engineer of the year for the entire LMCO organization. Not what you would expect from someone with his background, but gambling in hiring does pay off.) Frank showed me some memos that explained how he arrived at this result based on his work as well as that done by Paul Kalata (I have known Paul for a long time, but can't localize our first meeting to a particular date in time.), and I told him how I arrived at the same relationship. (Much of this performance analysis was used internally within the Naval community and was latter summarized in the internal memo entitled "The Working Engineers Guide To $\alpha - \beta$ and $\alpha - \beta - \gamma$ Filters" by Reifler and Solomon [17].) We both agreed that it would be difficult to extend the Z-transform approach to attack some problems that we thought needed to be solved in radar tracking. Frank and I discovered a common interest in theoretical physics which has led to a friendship that we have maintained over the years by occasional visits and reading and commenting on each others papers. I then went on to work on other radar related problems including designing a signal processor for a radar and a variety of other interesting problems.

I left Dahlgren for a while (... long complicated and not relevant except that I learned some more physics and mathematics as well as some personality limitations on what I was capable of handling), returned and went to work for a different organization that was concerned with firing missiles as well as radar related problems. The $\alpha - \beta$ filter reared its head again as I was trying to determine how the filter would respond to different input models of target dynamics that are included in the modeling process for sensor measurement such as, $x_m(k) = \frac{1}{2}aT^2k^2, = \sin(\omega kT)$, for example. This lead to a number of publications related to this problem [6], [7]. A simpler approach that might be useful as a classroom example is found in [9], [21]. Any constant gain filter has equations that can be written as

$$|x_s(k)\rangle = F\,|x_s(k-1)\rangle + G\,x_m(k). \tag{11}$$

$$|x_p(k+1)\rangle = \Phi\,|x_s(k)\rangle \tag{12}$$

where $x_m(k)$ is the deterministic measurement model. (An actual measurement is stochastic, with the stochastic element denoted as ˆ) Measurement associated with a sensor is represented as

$$\hat{x}_m(k) = x_m(k) + \hat{x}_n\,(k) \tag{13}$$

where $\hat{x}_n$ is the random component. The measurement model can be either a scalar or a vector. The F, D, G are matrices related to the form of the particular dimension of the constant gain filter. For the $\alpha - \beta$ filter, the matrices in (1) are specifically

$$|x_s(k)\rangle = \begin{bmatrix} x_s(k) \\ v_s(k) \end{bmatrix}, \tag{14}$$

$$|x_p(k)\rangle = \begin{bmatrix} x_p(k) \\ v_p(k) \end{bmatrix} \tag{15}$$

$$F_{\alpha-\beta} = \begin{bmatrix} 1-\alpha & (1-\alpha)T \\ -\frac{\beta}{T} & 1-\beta \end{bmatrix}, \tag{16}$$

$$G_{\alpha-\beta} = \begin{bmatrix} \alpha \\ \frac{\beta}{T} \end{bmatrix}, \tag{17}$$

$$\Phi_{\alpha-\beta} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}. \tag{18}$$

By continuing to take differences of $|x_s(k)\rangle$

$$\Delta^1\,|x_s(k)\rangle = |x_s(k)\rangle - |x_s(k-1)\rangle \tag{19}$$

until $\Delta^i x_m(k) =$ constant (for example $i = 2$ for the constant acceleration model $x_m(k) = \frac{1}{2}aT^2k^2$) or for periodic models until $x_m(k + \omega lT) = x_m(k)$, one gets a homogenous difference equation which be solved for any model of practical interest. In 1988, Frank and I were discussing the differences in our approaches to solving this problem when he invited me to attend a physics conference at the University of Maryland. I met a friend of Frank's who is a professor at Georgetown University in the mathematics department (Dr. Andrew (Andy) Vogt (Erdos number 2 so I have an Erdos number of 3) who is now the chair of the mathematics department.). I few months later we met again at another physics conference at George Mason University dedicated to work by David Bohm. Both Frank and I spent the weekend at Andy's condo. A few weeks later Andy called me and told me he was coming down with his wife and a friend to go hiking near where I lived, "Would I like to come?" His "friend" and I were married within six months of our first date. Andy and I make a point of getting together regularly, our latest paper, which appeared this May, is on the time-energy uncertainty relation in quantum mechanics.

In the early nineties, I worked primarily in the area of signal processing and Doppler radar, with another brief excursion into Doppler tracking with a colleague, Dr. William (Dale) Blair, who left Dahlgren about eight years ago. Dale is now a fellow of the IEEE. Dale decided to organize a short course at Dahlgren on the topic of multi-sensor tracking, data association, and other more complicated issues in tracking. The most comprehensive book on the subject at the time was by Dr. Yaakov Bar-Shalom [1]. Yaakov gave the short course that I attended. It remains the best short course that I have ever taken.

Yaakov continues to write books on tracking that are the most comprehensive in the field. They are a must read for anyone working in target tracking and estimation problems that are sensor related. During the course, the subject of the $\alpha-\beta-\gamma$ filter came up in an example. Kalata had shown that for the $\alpha-\beta$ filter, one could solve for α in terms of known quantities using the relationship between the filter coefficients. The tracking index

$$\Gamma^2 = \frac{\beta^2}{1-\alpha}, \tag{20}$$

allows one to relate the coefficients, α and β, to the sensor design parameters related to threat maneuverability and sensor noise. The variable Γ_K is the Kalata tracking index,

$$\Gamma_K^2 = \frac{T^4\sigma_a^2}{\sigma_m^2} = \Gamma^2. \tag{21}$$

The tracking index is a function of the assumed target maneuverability variance σ_a^2 (deviation from modeled behavior), radar measurement noise variance σ_m^2, and T the update interval. For the $\alpha-\beta-\gamma$ filter, the coefficients had been solved numerically since the tracking index appeared to give a sixth order polynomial equation that had to be solved. After class, I said that this equation could be reduced to a third order polynomial equation, which could then be solved exactly. After initial surprise, Yaakov asked to see the solution. After examining it, he encouraged me to publish it, which I did [8].

Subsequently, another problem of a similar nature came up which I solved; he published with accreditation to me in a latter book [2].

Yaakov is a wonderful reference source. If anything has been published in the area of track filtering, he is aware of it and many of the interesting things have been published by him or his former students. He is so prolific in the area of estimation, that one can talk about a Bar-Shalom number in the same sense that one talks about an Erdos number. He always knows where the best Thai restaurants are in any place a conference is being held. He and I always manage to spend some time talking about what history books we have been reading and what is going on in current affairs whenever we meet at tracking conferences. Through his books, lecturing, and papers, he has contributed more to the practical usage of Kalman filtering and its generalizations to sensors than Kalman ever did.

In 1993, my health collapsed so I no longer had the energy to both work and publish. This persisted for a couple of years, but my friend Andy took up the slack. He introduced me to the Washington Evolutionary System Society (WESS). I started attending their meetings and eventually gave some informal talks at their conferences. One member of WESS, George Farre, enjoyed a talk I gave on generalizations of the principle of superposition relative to the problem of emergence. (This idea arose out of the signal processing work I had been doing.) He invited me to give a talk on the subject at the ECHO conference in Germany. (Echo conferences are devoted the general area of emergence.) I became involved in ECHO attending the conferences over the years. This year, I gave a talk at a category theory conference affiliated with ECHO this year. As a consequence of this affiliation, I have met a number of well known mathematicians and physicists including a Nobel prize winner, who I occasionally correspond with about five sigma phenomena and the limitations of the scientific method.

For the next few years 1995-2000, I gradually started training people and publishing some things that came out of my training people on different aspects of target tracking problems. As a result I solved some questions Frank and I had raised during discussions with each other. Then I transferred to another department and have concentrated on theoretical aspects of estimation related to sensor fusion, radar signal processing, and other stochastic issues that are important in Naval applications. Whenever I have been asked to train someone in the basis of tracking, I have always returned to the $\alpha-\beta$ filter as part of the training process and usually find something new to help better understand some aspect of target tracking. Contrary to the modern tendency of viewing the $\alpha - \beta$ filter as being derivative, as a special case of the Kalman filter, I find it useful to think of it being entirely independent of the Kalman filter. Such concepts as noise reduction, filter coefficient selection criteria, bias response for the filters and relationships between the filter coefficients can be chosen without any reference to the Kalman filter. This is not the point of view of most authors, but is held by Reifler and Kalata. One of my last "students", Sunshine Smith-Carroll (she is a program manager now and I am doing some consulting work for her) have argued this in a couple of publications [10], [11]. In doing so, we have returned to my first problem—which I now have a better solution to which I present because it raises some new problems.

Given the matrix equations for the tracking filter, the covariance matrix of a first order system can be defined as [22](where $'$ denotes transpose)

$$\begin{aligned} P_k &= E\{(|\hat{x}_s(k)\rangle - E(|\hat{x}_s(k)\rangle)(\langle\hat{x}_s(k)| - E(\langle\hat{x}_s(k)|))\} \\ &= E\{(|\hat{x}_s(k)\rangle - |\overline{x}_s(k)\rangle)(\langle\hat{x}_s(k)| - \langle\overline{x}_s(k)|), \end{aligned}$$

where $E(|x_s(k)\rangle) = |\overline{x}_s(k)\rangle = F\,|\overline{x}_s(k-1)\rangle$. Note we assume that there is noise is input only, so

$$E(\hat{x}_m(k)) = 0 \tag{22}$$

and

$$E(\hat{x}_m^2(k)) = \sigma_n^2(k). \tag{23}$$

One can then calculate the covariance matrix P_k as

$$P_k \triangleq E(|\hat{x}_s(k)\rangle\,\langle\hat{x}_s(k)|) = E\{F\,|\hat{x}_s(k-1)\rangle\,\langle\hat{x}_s(k-1)|\,F' + \sigma_n^2(k)GG'\}. \tag{24}$$

The expected value is explicitly given as

$$P_k = FP_{k-1}F' + \sigma_n^2(k)GG'. \tag{25}$$

In steady state ($P_k = P_{k-1} = P$), so the covariance is in the form of a Lyapunov matrix equation

$$FPF' - P = -\sigma_n^2 GG', \tag{26}$$

which can be written as ($M = \sigma_n^2 GG'$)

$$P = FPF' + M. \tag{27}$$

This is a Lyapunov equation.

An approach [3] to solving a Lyapunov equation (F, M are known) is to define an anti-symmetric matrix S as

$$S = PF' - FP. \tag{28}$$

With $M = P - FPF'$, it is easy to show that S satisfies the following equation

$$FSF' - S = FM - MF' = B \tag{29}$$

which is another Lyapunov equation with one fewer unknown instead of the original equation since it is antisymmetric as is the matrix; $B = FM - MF'$. Now add these two equations together and after a few manipulations, we have a solution for P, namely that it is a solution to the equation

$$P = (F + I)^{-1}(S - M)(F^{'} - I)^{-1}. \tag{30}$$

Since the matrices P and M are symmetric, the matrices S and B are anti-symmetric ($S = -S'$, $B = -B'$). The fact that S is anti-symmetric simplifies our calculations for S. Calculation of the covariance can be considerably simplified by solving the Lyapunov equation for S versus solving the equations for the covariance matrix P directly. For a 2×2 matrix, one has to solve for one unknown instead of two; for a 3×3 matrix, one has to solve for three unknowns instead of six, etc. Thus reduction in the number of variables has considerable advantage for higher dimensional filters such as the $\alpha - \beta - \gamma$ filter. Solving the specific Lyapunov equation for the $\alpha - \beta$ filter gives the noise reduction ratios that have already been presented. Similar analysis can be used to solve for the transient noise reductions ratios as is demonstrated.

Uncertainty about threat behavior $\widetilde{w}(k)$ can be lumped into the predicted update as a maneuver model. Note $\widetilde{w}(k)$ is white process noise that is zero-mean which has a second moment, so this means

$$E(\tilde{w}(k)) = 0,\ E(\widetilde{w}(k)^2) = \sigma_w^2. \tag{31}$$

In matrix form, the predicted update has what is termed a **Maneuver Model**, which is given by

$$|\hat{x}_p(k+1)\rangle = \Phi\,|\hat{x}_p(k)\rangle + \Psi\tilde{w}(k), \tag{32}$$

The update procedures for the Maneuver model covariance obeys for the update procedure

$$T_k = \Phi T_{k-1}\Phi' + \Psi\sigma_w^2\Psi', \tag{33}$$

where

$$T_k \triangleq E(|\hat{x}_p(k)\rangle\langle\hat{x}_p(k)|) \tag{34}$$

There are two possible models for Ψ: **Model 1** where all the uncertainty is in the position

$$\Psi_1 = \begin{bmatrix} T^2/2 \\ 0 \end{bmatrix}, \tag{35}$$

and **Model 2** where the uncertainty is jointly in the position and velocity

$$\Psi_2 = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}. \tag{36}$$

Now for **Model 1**, ignoring all the details found in [10], so one arrives at the error covariance matrix

$$T^{(1)} = \sigma_w^2 \begin{bmatrix} T_x^1(0) & T_{xv}^1(0) \\ T_{xv}^1(0) & T_v^1(0) \end{bmatrix} \tag{37}$$

so the transient noise reduction ratio for the position and the noise reduction ratio for the velocity are

$$T_x^1(0) = \frac{(2-\alpha)(1-\alpha)^2T^2}{\beta D}, \tag{38}$$

$$T_v^1(0) = \frac{\alpha^2(2-\alpha) + 2\beta(1-\alpha)}{\beta D}. \tag{39}$$

where

$$D = \alpha(4 - 2\alpha - \beta). \tag{40}$$

The other approach to maneuver uncertainty is to incorporate uncertainty into both the position and velocity prediction components as in **Model 2**, .

$$T^{(2)} = \sigma_a^2 \begin{bmatrix} T_x^2(0) & T_{xv}^2(0) \\ T_{xv}^2(0) & T_v^2(0) \end{bmatrix} \tag{41}$$

The transient noise reduction ratio for position is

$$T_x^2(0) = \frac{(1-\alpha)^2T^4}{2\alpha\beta} \tag{42}$$

while the transient reduction ratio is

$$T_v^2(0) = \frac{T^2}{4}\left[\frac{2\alpha^2 - 3\alpha\beta + 2\beta}{\alpha\beta}\right]. \tag{43}$$

Knowledge of the transient and steady state noise reduction ratios allow one to determine the filter coefficient relationships $\beta = \beta(\alpha)$. The Jacobian of variables x and y with respect to variables u and v is defined as [14]

$$J(x,y;u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \left(\frac{\partial x}{\partial u}\right)_v & \left(\frac{\partial x}{\partial v}\right)_u \\ \left(\frac{\partial y}{\partial u}\right)_v & \left(\frac{\partial y}{\partial v}\right)_u \end{vmatrix} = \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_u - \left(\frac{\partial x}{\partial v}\right)_u \left(\frac{\partial y}{\partial u}\right)_v \tag{44}$$

(Note, since $x/y = x/y\,(u,v)$, when taking the partial with respect to one variable, the other variable is held constant, this is what the notation $\left(\frac{\partial x}{\partial u}\right)_v$ means. *Some mathematicians are now pulling their hair out.*) Jacobians are quite useful when one is considering functions that are joint functions of two variables. (Jaynes has made this observation in some unpublished notes on thermodynamics). A method for interpreting the Jacobian of two functions is to examine two contours $A(x,y) = const.$and $B(x,y) = const.$in the $x - y$ plane. With the $z-$axis pointing out of the plane, the Jacobian is interpreted as the $z-$component of the cross product of the gradient of A and B

$$J(A,B;x,y) = \left(\frac{\partial A}{\partial x}\right)_y \left(\frac{\partial B}{\partial y}\right)_x - \left(\frac{\partial A}{\partial y}\right)_x \left(\frac{\partial B}{\partial x}\right)_y = (\nabla A \times \nabla B)_z \tag{45}$$

This is equal to the area of the parallelogram with sides ∇A and ∇B. At the point where the Jacobian vanishes, the area of the parallelogram is zero. Then any infinitesimal change to A also holds B constant. This gives an explicit geometric interpretation of filter coefficient relationship as the tangent curve which holds the contours $B(\alpha,\beta)$ =constant and $A(\alpha,\beta)$ =constant for all possible values. The position error function **Model 1** is

$$A_p^{(1)}(\alpha,\beta) = \frac{2\alpha^2 + 2\beta - 3\alpha\beta}{\beta\alpha(4 - 2\alpha - \beta)}, \tag{46}$$

$$B_p^{(1)}(\alpha,\beta) = \frac{(2-\alpha)(1-\alpha)^2}{\beta\alpha(4 - 2\alpha - \beta)} \tag{47}$$

while the velocity error function **Model 1** is

$$A_v^{(1)}(\alpha,\beta) = \frac{2\beta^2}{\alpha(4-2\alpha-\beta)}, \tag{48}$$

$$B_v^{(1)}(\alpha,\beta) = \frac{\alpha^2(2-\alpha)+2\beta(1-\alpha)}{\beta\alpha(4-2\alpha-\beta)} \tag{49}$$

If we substitute these particular $A(\alpha,\beta)$ and $B(\alpha,\beta)$ into the Jacobian expression, by setting $J(A(\alpha,\beta), B(\alpha,\beta); \alpha,\beta) = 0$, gives the filter coefficient relationship

$$\beta = \frac{\alpha^2}{2-\alpha}, \tag{50}$$

which is the Benedict-Bordner relationship. The same things happens for the position criteria, so our formalism leads to the Benedict-Bordner relationship for both criteria. The position noise reduction ratio along with the new position transient error response derived from **Model 2**, give the joint transient position criteria functions

$$A_p^{(2)}(\alpha,\beta) = \frac{2\alpha^2+2\beta-3\alpha\beta}{\beta\alpha(4-2\alpha-\beta)}, \tag{51}$$

$$B_p^{(2)}(\alpha,\beta) = \frac{(1-\alpha)^2}{2\alpha\beta}. \tag{52}$$

Applying the Jacobian optimization criteria to these two functions gives the Kalata relationship

$$\beta = 2(2-\alpha) - 4\sqrt{1-\alpha}. \tag{53}$$

A number of other relationships are possible depending on what choice one makes for the scalar functions A and B. For most applications one wants to consider noise reduction as the one of the contours that contribute to the determination of $\beta = \beta(\alpha)$, but the "acceleration term" could be replaced with a variety of other possibilities. A variety of different filter coefficient relationships are possible once one realizes this. This approach outlined should work for any constant gain filter. It also points to the possibility that one can rethink some of the estimation problems in tracking in more geometric terms, perhaps I will return to this line of thought someday. Another observation is that the Jacobian is equivalent to the Poisson bracket in physics. If we didn't set the Jacobian equal to zero, but set it equal to a constant, then this procedure is the equivalent of quantization in physics. Does this mean anything? Time will tell.

I have engaged a bit in my physicist tendency, namely to over simplify and wax poetic about exactly solvable problems. Understanding the alpha-beta filter has been a constant source of learning and discovery, certainly a love over these many years. *I would like to thank my friends and collaborators (some of whom didn't act in this scene Ali and Hugh), with whom I have worked on different aspects of target tracking. In particular those who contributed to my understanding of "constant gain filters" over the years include: Terry Foreman, who started me down this path, Frank Reifler and Paul Kalata, who listened to when I talked from left field, Yaakov Bar-Shalom and Dale Blair who listened when I wasn't really sure what I was doing, Bill Murray and Sunshine Smith-Carroll who both did training assignments under me and trained me as much as I did them. Lastly, I thank Andy and especially Pat, who helped me stay on the path after some near disasters.*

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