ATEP SYS12525

Day 4

Radar Trackers and Applications for SAADS

http://www.jameskbeard.com

Topics For Today (1 of 2)

- Pulse Amplitude Estimation -- An Unusual Case
- Modeling The Measurements
  - Chirped Pulses
  - Bin Splitting and Monopulse
  - Centroiding
- INS Errors in Trackers
- Correlated Measurement Noise

Topics For Today (2 of 2)

- Simulating Trackers
- Monte Carlo Simulation
- Variance Reduction
- Interactive Multiple Models
- Implementing Batch Estimators
- Tensors in Sensor Systems Analysis
- Technology Trends
- Beard’s Law

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Topic 22: Pulse Amplitude Estimation

Sensor Systems Engineering for the 21st Century

Pulse Amplitude Estimation

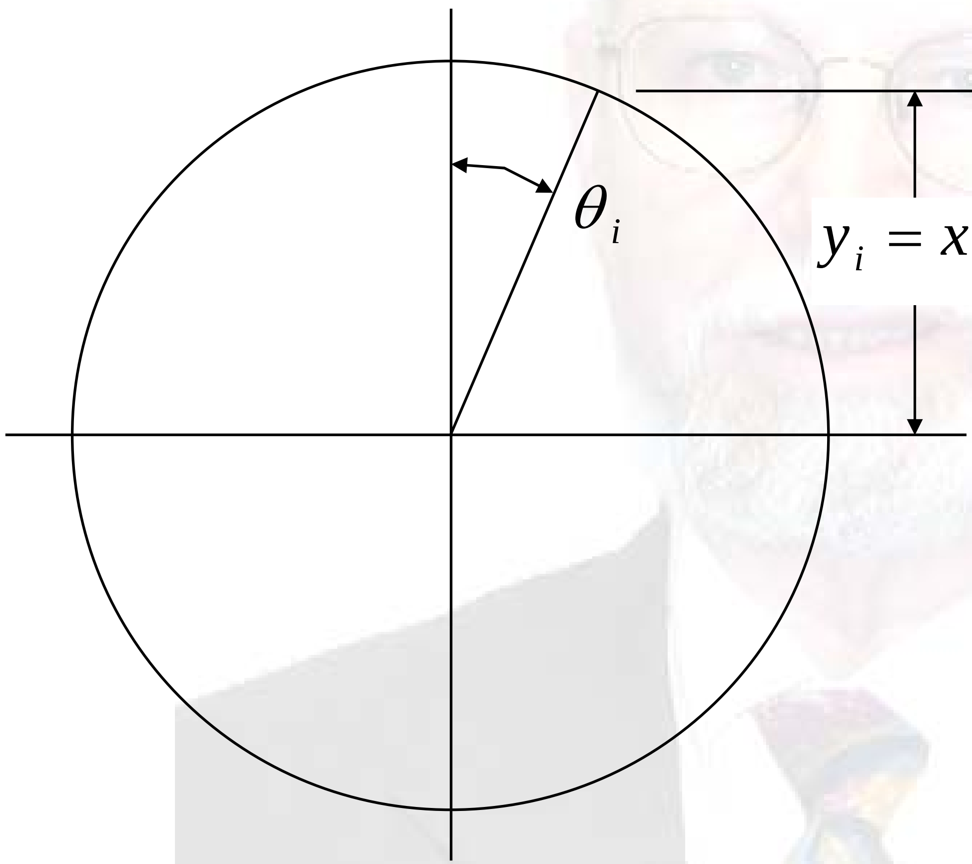
- Example: Estimate Pulse Shape from Digitized Pulses in a Burst
- Magnetron Transmitter -- Phase Incoherent Pulse to Pulse
- Data is Collection of Samples
  - Real Data Only – No Quadrature Demodulation
  - Phases are Random and Uncorrelated
- Data Vector $\underline{y}$ is

$$\underline{y} = \begin{bmatrix} x_a \cdot \cos(\omega_0 \cdot t_k + \phi_1) \\ \vdots \\ x_a \cdot \cos(\omega_0 \cdot t_k + \phi_N) \end{bmatrix}$$

Problem Description

- Quantity to be Estimated is
  - Pulse Amplitude x_a at Time t_k
  - Parameters are Constant Over Data
- Measurement Noise
  - Random Angles f_i, $1 \le i \le N$
  - Amplitude Measurement Noise Ignored Here

Estimate the Radius of the Circle



$$p(\theta)\begin{cases} = \dfrac{1}{2\pi}, & |\theta| < \pi \\ = 0, & |\theta| \ge \pi \end{cases}$$

$$y_i = x_a \cdot \cos(\theta_i)$$

$$\begin{aligned} P(y_i \le Y) &= P(\theta_i \ge \theta(Y)) \\ &= 1 - \frac{1}{\pi}|\theta(Y)| \\ &= 1 - \frac{1}{\pi}\cos^{-1}\left(\frac{Y}{x_a}\right) \end{aligned}$$

Likelihood Function

- For the Amplitude Measurement at Each Point in Time

$$p\left(y_i \middle| x_a\right) = \frac{1}{\pi} \cdot \frac{1}{\sqrt{x_a^2 - y_i^2}}$$

- For The Data Vector $\underline{y}$

$$p\left(\underline{y} \middle| x_a\right) = \frac{1}{\pi^N} \cdot \prod_{i=1}^{N} \frac{1}{\sqrt{x_a^2 - y_i^2}}$$

The Likelihood Equation

- Likelihood Function is

$$l(x_a) = -N \cdot \ln(\pi) - \frac{1}{2} \cdot \sum_{i=1}^{N} \ln\left(x_a^2 - y_i^2\right)$$

- Gradient is

$$\frac{\partial l(x_a)}{\partial x_a} = -\sum_{i=1}^{N} \frac{x_a}{x_a^2 - y_i^2}$$

Maximum Likelihood Estimate

- Properties of Log Likelihood Equation
  - Exists Only for $x_a > Max(y_i)$
  - Monotonic in x_a - Increases as x_a Decreases
  - Singular at $x_a = Max(y_i)$
  - Unbounded Above as x_a Approaches Singularity
- Maximum Likelihood Estimate of x_a is

$$\hat{x}_a = \underset{i}{Max}\left(y_i\right)$$

Statistics of Phases

- Assumptions
  - Phase Angles θ_i are Uniformly Distributed
  - Number N of Phase Angles is Large
- Mean Density of Phases is N/2π
- Two-Sided Probability of Phase from 0, Computed in Δθ Increments

$$1 - P(\theta \le k \cdot \Delta\theta) \approx \left(1 - \frac{N}{\pi} \cdot \Delta\theta\right)^k$$

Statistics of Estimate

- Variable Change k·Δθ=Θ, Take Limit k → ∞

$$1-P(\theta\le\Theta)\approx\left(1-\frac{N}{\pi}\cdot\frac{\Theta}{k}\right)^{k}\rightarrow\exp\left(-\tfrac{N}{\pi}\cdot\Theta\right)$$

- Minimum Phase Angle from 0 is Poisson

$$p(\Delta\theta\le\Theta)=1-\exp\left(-\tfrac{N}{\pi}\cdot\Theta\right)$$

- PDF of Largest Cosine

$$P\left(LUB\left(\cos(\theta_i)\right)\le X\right)=1-P\left(\theta_i\le\cos^{-1}(X)\right)$$

$$=\exp\left(-\frac{N}{\pi}\cdot\cos^{-1}(X)\right)$$

Statistics of Estimate (Continued)

- Probability Density Function is

$$p\left(\frac{\hat{x}_a}{x_a}\right) = \frac{N}{\pi} \cdot \frac{\exp\left(-\frac{N}{\pi} \cdot \cos^{-1}\left(\frac{\hat{x}_a}{x_a}\right)\right)}{\sqrt{1-\left(\frac{\hat{x}_a}{x_a}\right)^2}}$$

- Better to use the simpler PDF
  - Define Confidence Thresholds
  - Analyze Properties of Estimate

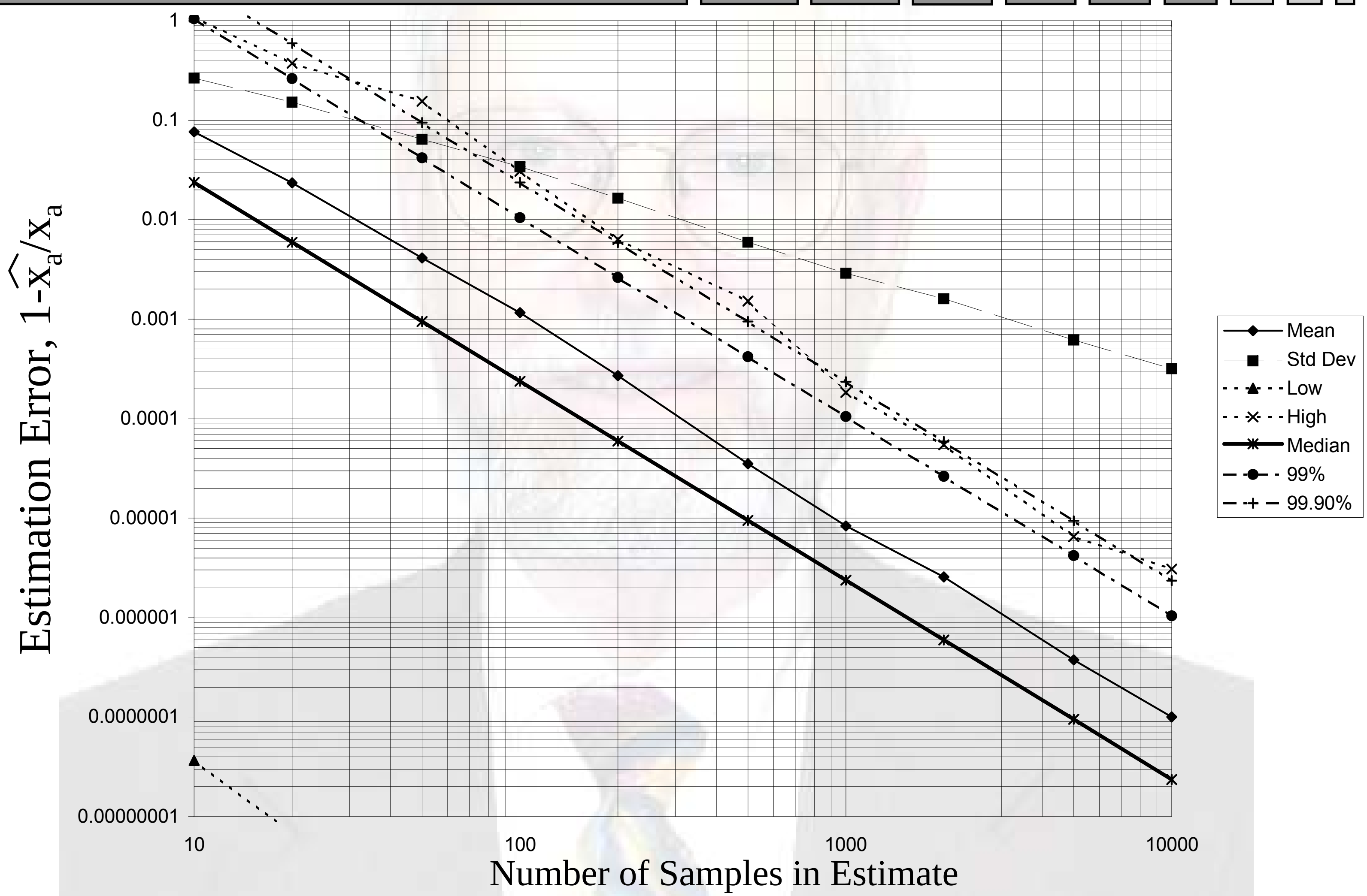
Accuracy of Estimate

- Use PDF to Tie Together
  - Specified Minimum Value X of Estimate $\underline{x}_a/\widehat{\underline{x}}_a$
  - Probability P_C That Estimate Exceeds Minimum
  - Number N of Data Points Needed

$$P_C = 1 - \exp\left(-\frac{N}{\pi} \cdot \cos^{-1}(X) \right)$$

$$\frac{x_a - \hat{x}_a}{x_a} \approx \frac{1}{2} \cdot \left(\frac{\pi \cdot \ln(1 - P_C)}{N} \right)^2$$

1000 Run Monte Carlo



Very Unusual Properties

- Decrease in Dispersion of Estimate with N
  - For Averaging Type Estimators is $1/N^{1/2}$
  - For This Estimator, Dispersion Decreases
    - » Mean (accurately predicted as $(\pi/N)^2$), Centiles as $1/N^2$
    - » Standard Deviation (accurately predicted as π/N) as $1/N$
    - » Behavior Verified with 1000 Sample Monte Carlo
    - » Maximum Value of 1000 Tries Tracks 99.9% Confidence
- Why?
  - Least Squares Estimators Average the Data Variances
  - This Estimator has PDF with Singularity at MLE Solution
  - Estimate Error is: 1 - Cosine(Poisson Distributed Variable)

Summary

- Mapping Property
  - Holds for Matrix Estimators
  - Applies for MLE
  - Useful in Keeping Things Simple
- Non-Gaussian Measurements
  - MLE Still Works
  - The Rules are Problem-Dependent

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Topic 23: Modeling the Measurements

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Chirped Pulses

- Used in
  - BMEWS, Pave Paws
  - GBR
  - Most Conventional SAR
  - Other
- Measure Linear Combination of
  - Doppler
  - Range

Processing Chirped Pulses

- Dechirp and FFT
  - Measure Dechirped Offset from Center Frequency

$$f_{meas} = f_0 - \frac{2 \cdot \dot{R}}{\lambda} + \frac{2 \cdot \dot{f} \cdot \Delta R}{c}$$

  - Doppler Measured Directly
  - Range Measured through Chirp Rate
- Ambiguous Measurement
  - Of Doppler
  - Of Range

Measurements from Chirped Pulses

- Ambiguous Measurement of Range

$$R_{Ambig} = R - \frac{f_0}{\dot{f}} \cdot \dot{R}$$

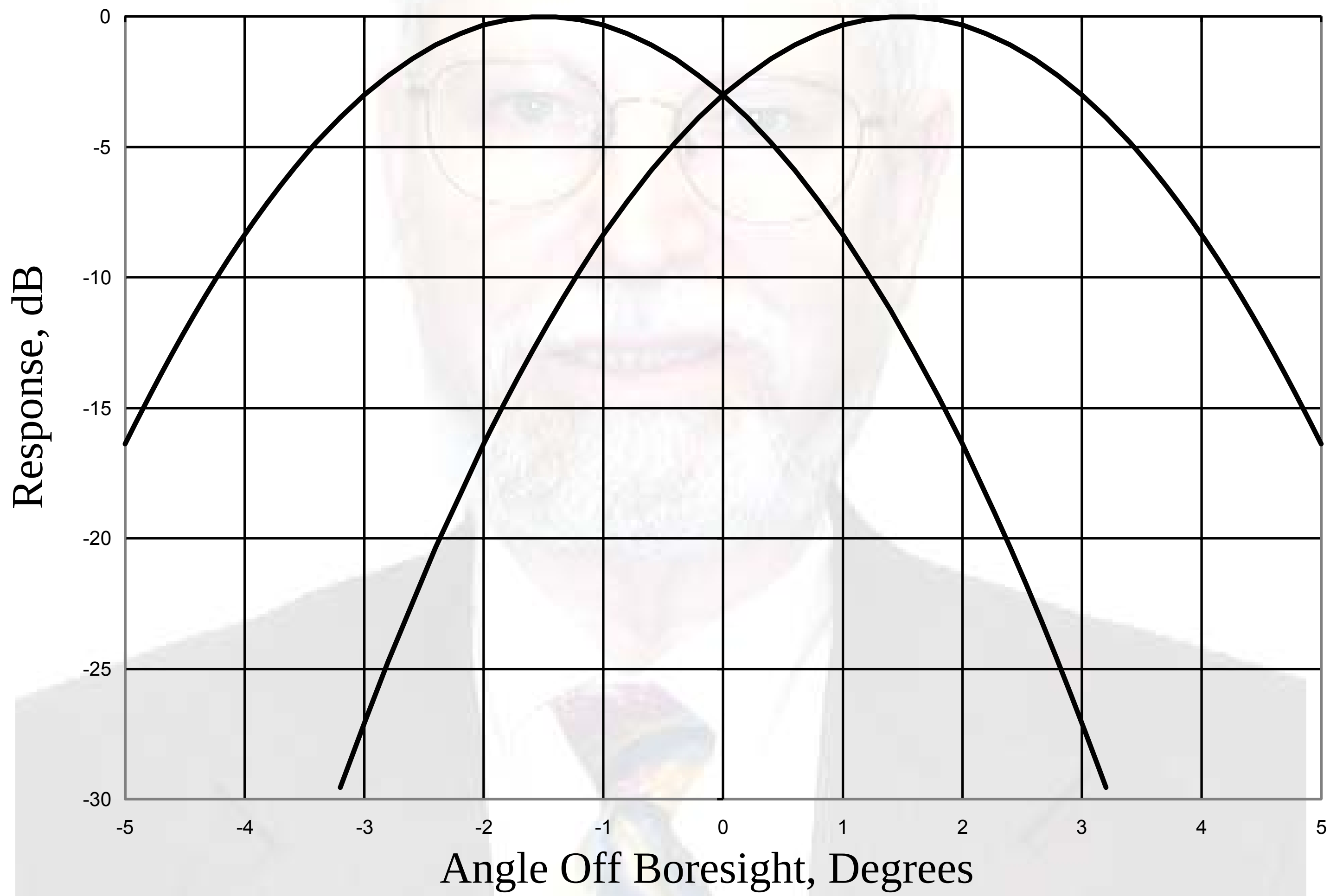
- Ambiguous Measurement of Range Rate

$$\dot{R}_{Ambig} = \dot{R} - \frac{\dot{f}}{f_0} \cdot \Delta R$$

The Bin Splitting Problem

- Applications to Systems Analysis
  - Amplitude and Phase Monopulse
  - Doppler Estimation from Energy in Multiple FFT Bins
  - Range Estimation from Energy in Multiple Range Bin Processors or Stretch Processing Bins
  - Any Incoherent Two Bin Measurement

Angle Estimation Monopulse



An Estimation Theory Problem

- Model
  - Parabola Fit to Main Lobe Shape

$$y_i(\theta) = E_s \cdot \exp\left(-2 \cdot \ln 2 \cdot \left(\frac{\theta \pm \theta_1}{\theta_3} \right)^2 \right) + v_i, \;\; i = 1,2$$

  - State Vector

$$\underline{x} = \begin{bmatrix} \theta \\ E_s \end{bmatrix}$$

- Solution: Nonlinear and Messy

Variable Change

- Variable Changes

$$e_i = \ln(E_i) = e_s - a \cdot (\theta \pm \theta_1)^2 + \frac{v_i}{E_i}$$

$$e_s = \ln(E_s),\ a = \frac{2 \cdot \ln 2}{\theta_3^2}$$

- Map the Problem Statement
  - Linearize the Estimator
  - Nonlinear Transformation
  - Valid for High SNR

Mapped Problem Statement

$$\underline{y}' = \begin{bmatrix} e_1 - e_2 \\ e_1 + e_2 \end{bmatrix} = \begin{bmatrix} 4 \cdot a \cdot \theta_1 \cdot \theta \\ 2 \cdot \left(e_s - a \cdot \left(\theta^2 + \theta_1^2 \right) \right) \end{bmatrix} + \begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix}$$

$$R' = \frac{\sigma_n^2}{\left(E_1 \cdot E_2 \right)^2} \begin{bmatrix} E_2^2 + E_1^2 & E_2^2 - E_1^2 \\ E_2^2 - E_1^2 & E_2^2 + E_1^2 \end{bmatrix}$$

$$\underline{x}' = \begin{bmatrix} \theta \\ e_s \end{bmatrix}$$

Mapped MLE Problem

- Log Likelihood Function

$$L(\underline{x}) = -\ln(2\pi) - \ln(|R'|)$$

$$-(\underline{y}' - \underline{h}(\underline{x}'))^T \cdot R'^{-1} \cdot (\underline{y}' - \underline{h}(\underline{x}'))$$

- Essential Matrices

$$H = \begin{bmatrix} 4 \cdot a \cdot \theta_1 & 0 \\ -4 \cdot a \cdot \theta & 2 \end{bmatrix}$$

$$R'^{-1} = \frac{1}{4\sigma_n^2} \cdot \begin{bmatrix} E_1^2 + E_2^2 & E_1^2 - E_2^2 \\ E_1^2 - E_2^2 & E_1^2 + E_2^2 \end{bmatrix}$$

MLE Solution

- States

$$\hat{\theta} = \frac{\theta_3^2}{4 \cdot \ln 2 \cdot \theta_1} \cdot \frac{e_1 - e_2}{2}$$

$$\hat{e}_s = \frac{e_1 + e_2}{2} + 2 \cdot a \cdot \hat{\theta}^2$$

- Shorthand for Covariance Equations to Follow

$$\Sigma = E_1^2 + E_2^2, \ \Delta = E_1^2 - E_2^2$$

$$SNR = \frac{E_s^2}{\sigma_n^2}$$

Covariances

- Inverse Covariance, or Information Matrix

$$P^{-1} = \frac{4}{\sigma_n^2} \cdot \begin{bmatrix} a^2 \cdot \left(\Sigma \cdot \theta_1^2 - 2 \cdot \Delta \cdot \theta_1 \cdot \hat{\theta} + \Sigma \cdot \hat{\theta}^2 \right) & -2 \cdot a \cdot \left(\Sigma \cdot \hat{\theta} - \Delta \cdot \theta_1 \right) \\ -2 \cdot a \cdot \left(\Sigma \cdot \hat{\theta} - \Delta \cdot \theta_1 \right) & \frac{\Sigma}{4} \end{bmatrix}$$

- Covariance Matrix

$$P = \frac{\sigma_n^2}{16 \cdot a^2 \cdot \theta_1^2 \cdot E_1^2 \cdot E_2^2} \cdot \begin{bmatrix} \Sigma & 2 \cdot a \cdot \left(\Sigma \cdot \hat{\theta} - \Delta \cdot \theta_1 \right) \\ 2 \cdot a \cdot \left(\Sigma \cdot \hat{\theta} - \Delta \cdot \theta_1 \right) & 4 \cdot a^2 \cdot \left(\Sigma \cdot \theta_1^2 - 2 \cdot \Delta \cdot \theta_1 \cdot \hat{\theta} + \Sigma \cdot \hat{\theta}^2 \right) \end{bmatrix}$$

Variances of Estimates

- Variance of Angle Estimate

$$\mathrm{Var}\left(\hat{\theta}\right) \approx \frac{1}{SNR} \cdot \frac{1 + 2 \cdot a \cdot \left(\theta_1^2 + \hat{\theta}^2\right)}{8 \cdot a^2 \cdot \theta_1^2}$$

- Variance of Amplitude Estimate
  - May Be Needed for SNR Estimate Validation
  - Complex Equation

Variance of Amplitude Estimate

$$\mathrm{Var}(e_s) = \frac{EXP}{4 \cdot SNR \cdot \theta_1^2} \cdot \left(\theta_1^2 \cdot S - 2 \cdot \theta_1 \cdot \hat{\theta} \cdot D + \hat{\theta}^2 \cdot S\right)$$

$$EXP = \exp\left(4 \cdot a \cdot \left(\theta_1^2 + \hat{\theta}^2\right)\right)$$

$$S = \exp\left(-2 \cdot a \cdot \left(\hat{\theta} - \theta_1\right)^2\right) + \exp\left(-2 \cdot a \cdot \left(\hat{\theta} + \theta_1\right)^2\right)$$

$$D = \exp\left(-2 \cdot a \cdot \left(\hat{\theta} - \theta_1\right)^2\right) - \exp\left(-2 \cdot a \cdot \left(\hat{\theta} + \theta_1\right)^2\right)$$

Topics for Discussion

- What happens when E_s is complex?
- What is the implication of Swirling I targets?
- How is this result related to amplitude monopulse?
  - Hint: $S = y_1 + y_2,\ D = y_1 - y_2,\ \dfrac{D}{S} = \dfrac{y_1 - y_2}{y_1 + y_2}$

$$y_i = E_s \cdot \left(1 + 2 \cdot \ln 2 \cdot \left(\frac{\theta \pm \theta_1}{2\theta_3} \right)^2 \right)$$

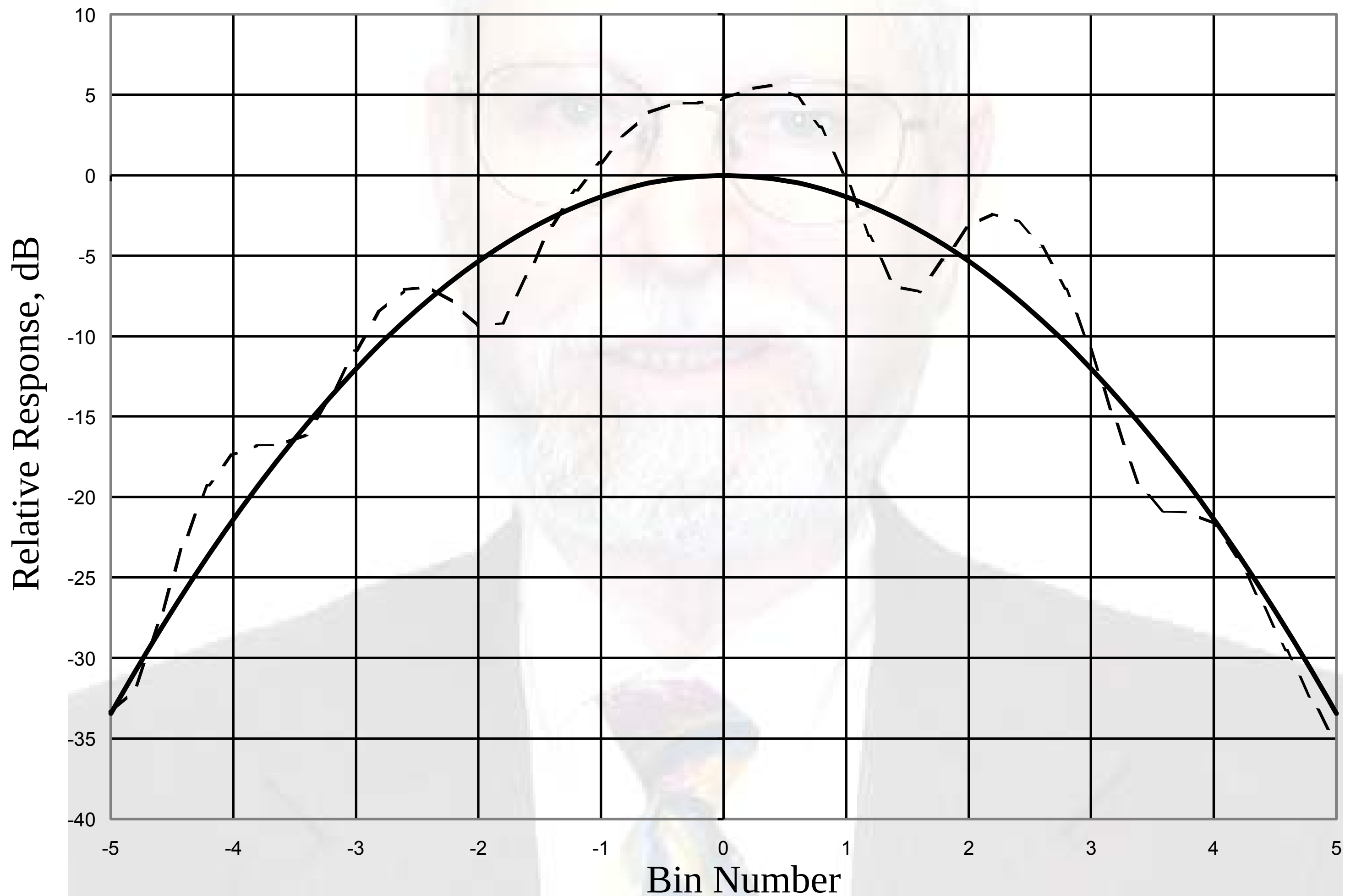
Discussion (Continued)

- What About Range Bins
  - With FM?
  - With Range Gating?
- What are the Measurement Noise Statistics when Splitting is Not Done?
- How About Target Returns Spread Across Many Bins?
  - Hint: Centroid

Centroiding

- Strong Signals
  - FFT Bins are Spread by Windowing
  - Bin Separation Not Affected by Windowing
  - Strong Signals Often Seen in Several Bins
- Complex Signals
  - Signal Bandwidth Exceeds FFT Bin Separation
  - Amplitude Glints from Bin to Bin

Scenarios



Relative Response, dB
10
5
0
-5
-10
-15
-20
-25
-30
-35
-40
-5
-4
-3
-2
-1
0
1
2
3
4
5
Bin Number

Centroiding Technique

- Very Commonly Used
- *Ad Hoc* Suboptimal Technique
- Define
  - Estimate of Center Frequency
  - Estimate of Spectral Spreading of Signal
  - Accuracy of Center Frequency Estimate
- The Equations Are ...

Center Frequency Estimate

- Center Frequency Estimate is Energy Weighted Mean

$$\hat{f}_c = \frac{\sum_i i \cdot W \cdot P_i}{\sum_i P_i}$$

  - Where
    - » W is the Bin Width
    - » P_i is Signal and Noise Energy in Bin i
- Variance Propagation Equation Defines Accuracy

Accuracy of Estimate

- Energy in Bin j

$$P_i = \left(v_i + n_i\right)^2, \ \left\langle n_i^2 \right\rangle = \sigma^2$$

- Sensitivity to Noise in Bin j

$$\frac{\partial \hat{f}_c}{\partial n_j} = \frac{2 \cdot \left(v_j + n_j\right) \cdot \left(j \cdot W - \hat{f}_c\right)}{\sum_i P_i}$$

- Variance of Estimate is

$$\operatorname{Var}\left(\hat{f}_c\right) = \sum_j \left(\frac{\partial \hat{f}_c}{\partial n_j}\right)^2 \cdot \sigma^2 = 4 \cdot \sigma^2 \cdot \frac{\sum_j \left(j \cdot W - \hat{f}_c\right)^2 \cdot P_j}{\sum_j P_j}$$

Variable Changes

- Total Power

$$P = \sum_i P_i$$

- A "Variance" or Mean Squared Dispersion

$$V = \frac{\sum_j \left(j \cdot W - \hat{f}_c \right)^2 \cdot P_j}{P}$$

- Final Form is

$$Var\left(\hat{f}_c \right) = 4 \cdot \sigma^2 \cdot V$$

For Use with Trackers

- Rule of Thumb

$$\mathrm{Var}\left(\hat{f}_c\right) \approx \frac{0.8}{SNR} \cdot (\text{Doppler Extent})^2$$

- Beginning Approximation for Tracker Design

$$\mathrm{Var}\left(\hat{f}_c\right) \approx \left(\frac{W}{2}\right)^2 + \frac{(\text{Doppler Extent})^2}{SNR}$$

- Give-Up Case

$$\mathrm{Var}\left(\hat{f}_c\right) \approx \frac{1}{12} \cdot (\text{Doppler Extent})^2$$

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Topic 24: INS Errors in Trackers

Sensor Systems Engineering for the 21st Century

Use of INU in Tracker

- Measurement model is actually a function of airframe position $\underline{p}$ as well as target position $\underline{x}$
- Airframe position data must be available to compute expected measurements
- INU Errors Map to Measurement Errors

Propagation of INU Errors

- Measurement Equation

$$\underline{y} = \underline{h}\left(\underline{p}, \underline{x}\right) + \underline{v}$$

- Propagation of INS Covariance to Measurement Covariance

$$R_{EQ} = Cov\left\{\underline{v}\right\} + H_P \cdot P_P \cdot H_P^T$$

- Details in Raytheon IDC 3453
- Significance in System Design
  – Measurement of Target Angles
  – Uninstrumented Jitter in EO/IR Systems

INU Errors

- Source of Errors in Tracker
- System Engineering Process:
  - Compute INU Errors
  - Compare to Other Tracker Errors
    - » If Small, Neglect or Approximate
    - » If Significant, Incorporate and Model
- INU Errors
  - Mapping to Tracker is Simple
  - Modeling in Tracker is Complex

Mapping INU Errors

- Measurement model

$$\underline{y} = \underline{h}\left(\underline{p}, \underline{\dot{p}}, \underline{\theta}, \underline{x}\right) + \underline{v}$$

- Airframe position data p must be available to compute expected measurements
- Own-Ship Velocities Needed for Range Rates
- Body Angles Needed for Computed Target Angles
- INU Errors Map to Measurement Errors

An Approach for Modeling INS Errors

- Measurement Model

$$\underline{y} = \underline{h}\left(\underline{p}, \underline{\dot{p}}, \underline{\theta}, \underline{x}\right) + \underline{v}$$

- Propagation of Covariance

$$R = Cov\left\{\underline{v}\right\} + H_P \cdot P_P \cdot H_P^T + H_\theta \cdot P_\theta \cdot H_\theta^T$$

- Need Position, Body Angles, and Rates
- Details from Raytheon IDC 3453 Follow

Example of INU Error Mapping: Phase Monopulse

- Measured Phase Angle over Antenna Baseline b Along Line of Sight $\underline{u}_r$

$$y = \frac{2\pi}{\lambda} \cdot \underline{b}^T \cdot \underline{u}_r + v$$

- Covariance Mapping Equation Requires

$$\frac{\partial \underline{b}}{\partial \underline{\theta}} = S_b \cdot V$$

$$S_b = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & \sin\gamma \\ 0 & -\cos\phi & -\sin\phi \cdot \cos\gamma \\ 0 & \sin\phi & -\cos\phi \cdot \cos\gamma \end{bmatrix}$$

INU Noise Mapping

- Mapped Noise Measurement

$$y = \frac{2\pi}{\lambda} \cdot \underline{b}^T \cdot \underline{u}_r + v + v_\theta$$

$$\mathrm{Var}\left\{v_\theta\right\} = \underline{d}^T \cdot R_\theta \cdot \underline{d}$$

$$R_\theta = Cov\left\{\underline{\theta}\right\}$$

$$\underline{d} = V^T \cdot \left(\tfrac{2\pi}{\lambda} \cdot \underline{b} \times \underline{u}_r\right)$$

- Other Mappings are Simpler

Euler Angles

- Roll, Pitch and Yaw

$$\underline{\theta} = \begin{bmatrix} roll \\ pitch \\ yaw \end{bmatrix} = \begin{bmatrix} \phi \\ \gamma \\ \psi \end{bmatrix}$$

- Order of Rotations
  - Convention is the Euler Sequence
  - Inertial to Body – Yaw, Pitch, Roll
  - Body to Inertial – Roll, Pitch, Yaw

Quaternion from Rotation Matrix

- Quaternion is (See Quaternion Report, P. 31)

$$q = \cos\left(\frac{\theta}{2}\right) + \underline{u} \cdot \sin\left(\frac{\theta}{2}\right)$$

- Antisymmetrical Part of A is

$$\frac{1}{2}\left(A - A^T\right) = 2 \cdot \cos\left(\frac{\theta}{2}\right) \cdot \sin\left(\frac{\theta}{2}\right) \cdot S_u = \sin(\theta) \cdot S_u$$

- S_u times Symmetrical Part of A is

$$S_u \cdot \frac{1}{2} \cdot \left(A + A^T\right) = \cos(\theta) \cdot S_u$$

Rotation Matrix from Quaternion

- Rotation Matrix A is

$$A = \cos(\theta) \cdot I + \sin(\theta) \cdot S_u + \left(1 - \cos(\theta)\right) \cdot \underline{u} \cdot \underline{u}^T$$

- Definition of S_u is

$$S_u = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

INU Errors Are Correlated

- INU Is
  - A Schuler Loop (Gelb, p. 56)
  - A Kalman Filter (Gelb, pp. 116-119)
  - GPS or Doppler Radar Aided
- INU Errors are Combination of
  - Kalman Filter Errors
  - GPS Errors
- Can Be Modeled by Markov Process
- Result: Correlated Measurement Noise

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Topic 25: Correlated Measurement Noise

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Correlated Measurement Noise

- Definition: Noise produced by a Markov process
- In Practice: Noise properly modeled as produced by a Markov process
- Arises When Input Bandwidth is Oversampled
- Effects
  - Reduces performance
  - Makes tuning difficult
  - Reduces robustness

References on Correlated Measurement Noise

- Gelb, pp 133-136, 332-335.
- Estimation and Tracking: Principles, Techniques, and Software, Yaakov Bar-Shalom and Xiao-Rong Li, YBS (ISBN 0-9648312-0-1) (1995)
- Applied Optimal Control, A. Bryson and Y.C. Ho, Hemisphere (1975)
- Estimation Theory with Applications to Communications and Control, McGraw-Hill (1971)
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- Stochastic Models, Estimation and Control, Vols. II, III, P.S. Maybeck, Academic Press (1982)

First Try: Augmented State Vector

- Model

$$\dot{\underline{x}} = F \cdot \underline{x} + G \cdot \underline{w},\ \mathrm{Cov}\{\underline{w}\} = Q \ \text{(System)}$$

$$\underline{z} = H \cdot \underline{x} + \underline{v},\ \mathrm{Cov}\{\underline{v}\} = R \ \text{(Measurement)}$$

$$\dot{\underline{v}} = E \cdot \underline{v} + \underline{w}_1,\ \mathrm{Cov}\{\underline{w}_1\} = Q_1 \ \text{(Measurement noise)}$$

- Augmented Case

$$\underline{x}' = \begin{bmatrix} \underline{x} \\ \underline{v} \end{bmatrix},\ F' = \begin{bmatrix} F & 0 \\ 0 & E \end{bmatrix},\ G' = \begin{bmatrix} G & 0 \\ 0 & I \end{bmatrix},\ Q' = \begin{bmatrix} Q & 0 \\ 0 & Q_1 \end{bmatrix}$$

$$H_2' = \begin{bmatrix} H & I \end{bmatrix}$$

Problems With Approach

- Measurement Noise is Zero
- Discrete Kalman Gain May Exist

$$K' = \tilde{P}' \cdot H'^{T} \cdot \left(H' \cdot \tilde{P}' \cdot H'^{T} \right)^{-1}$$

- Updated Inverse Covariance is Singular

$$P' = \tilde{P}' - \tilde{P}' \cdot H_2'^{T} \cdot \left(H_2' \cdot \tilde{P}' \cdot H_2'^{T} \right)^{-1} \cdot H_2' \cdot \tilde{P}'$$

$$P' \cdot H_2' = 0$$

- Square Root Approaches Not Applicable

Feasible Approach: Measurement Differencing

- State and Measurement Models

$$\underline{x}_+ = \Phi \cdot \underline{x} + G \cdot \underline{w},\ \ \mathrm{Cov}\{\underline{w}\} = Q$$

$$\underline{y} = H \cdot \underline{x} + \underline{v}$$

$$\underline{v}_+ = B \cdot \underline{v} + \underline{w}_1,\ \ \mathrm{Cov}\{\underline{w}_1\} = Q_1$$

- Prewhiten Measurements

$$\underline{z} = \underline{y}_+ - B \cdot \underline{y} = H_2 \cdot \underline{x} + \underline{v}_2$$

$$H_2 = H_+ \cdot \Phi - B \cdot H$$

$$\underline{v}_2 = H_+ \cdot G \cdot \underline{w} + \underline{w}_1$$

More Correlations

- Measurement and Plant Noise Correlated

$$C = \text{Exp}\left\{G \cdot \underline{w} \cdot \underline{v}_2^T\right\} = G \cdot Q \cdot G^T \cdot H_+^T$$

- Modify System Dynamics

$$\underline{x}_+ = \Phi \cdot \underline{x} + G \cdot \underline{w} + D \cdot \left(\underline{z} - H_2 \cdot \underline{x} - \underline{v}_2\right)$$

$$G = C \cdot R_2^{-1}, \;\; R_2 = H_+ \cdot G \cdot Q \cdot G^T \cdot H_+^T + Q_1$$

- New System Dynamics

$$\underline{x}_+ = \left(\Phi - D \cdot H_2\right) \cdot \underline{x} + D \cdot \underline{z} + G \cdot \underline{w} - D \cdot \underline{v}_2$$

Summary

- Correlated Process Noise Treated by
  - Measurement Differencing to Whiten Measurement Noise
  - Modification of States to Decorrelate Whitened Measurement Noise and Plant Noise
- Issues
  - H_+ Requires Next Value of State Vector
  - B is Complicated for Accurate Modeling of INS Errors

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Topic 26: Simulating Trackers

Sensor Systems Engineering for the 21st Century

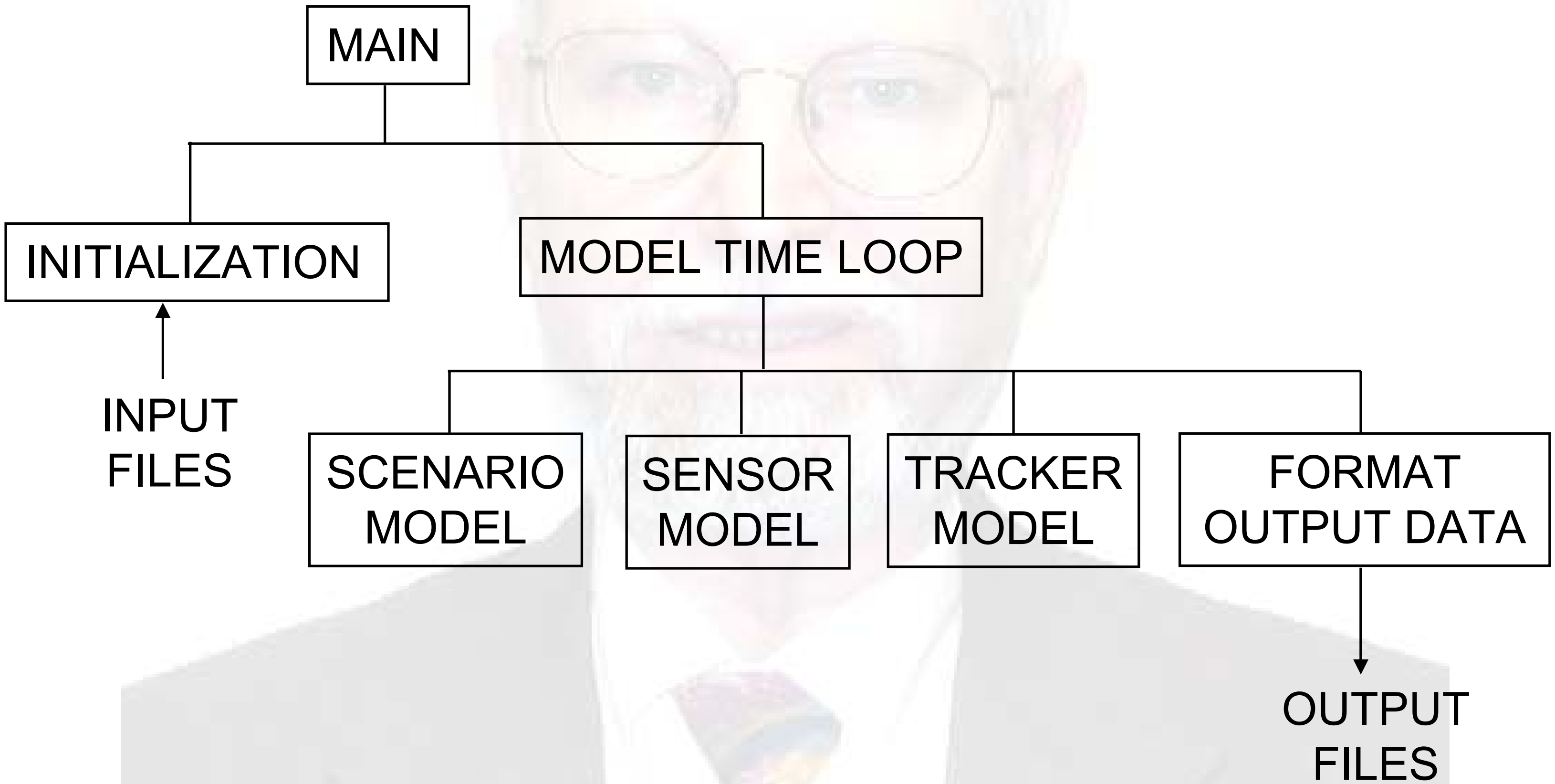
Simulating Trackers

- Requirements of Simulation
  - Support Development of Tracker Modules
  - Exercise Tracker Modules
  - Provide Basis for Monte Carlo Simulation
- Simulation Modules
  - Simulator -- Define Time Variables, Outputs
  - Scenario Generator -- Define Target Path
  - Tracker -- Build Tracker(s)
  - Output -- Write Output Data

Simulation Architecture

- Main Program
  - Call Initialization Modules
    - » Reads input files
    - » Converts data and stores in global memory
  - Initializes Simulation Log File
  - Calls Simulation
- Simulation Program
  - Runs Target Scenario Model
  - Runs Radar and Tracker Module
  - Writes Output Files
- Lower Level Modules
  - Actual Numerical Models
  - Tracker Modules
  - Utilities

Simulation Module Hierarchy



Types of Simulations

- Emulative Level
  - Emulates Actual Hardware Functions
  - Simulates Raw Input Data
    - » Signal Processor Output for Trackers
    - » A to D Output for Signal Processor Plus Tracker
    - » Random Number Generator Simulates Noise Effects
- Prediction Modeling
  - Used for RF Portion of Radar
  - Used for Entire System in Mission Effectiveness Models
  - Probabilities Implemented with Random Number Generator

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Topic 27: Monte Carlo Simulation

Sensor Systems Engineering for the 21st Century

Functions of Monte Carlo Analyses

- Detection and Diagnosis of Problems
  - Extreme Values
  - Association Statistics
- Tracker Tuning
  - Verification of Performance
    - Biases
    - Variances and Covariances
  - Finding Optimal Adaptive Plant Noise Coefficients
  - Tuning Association Thresholds
- Proving Point Designs
  - Performance Meets Requirements
  - Robustness Limits Meet Mission

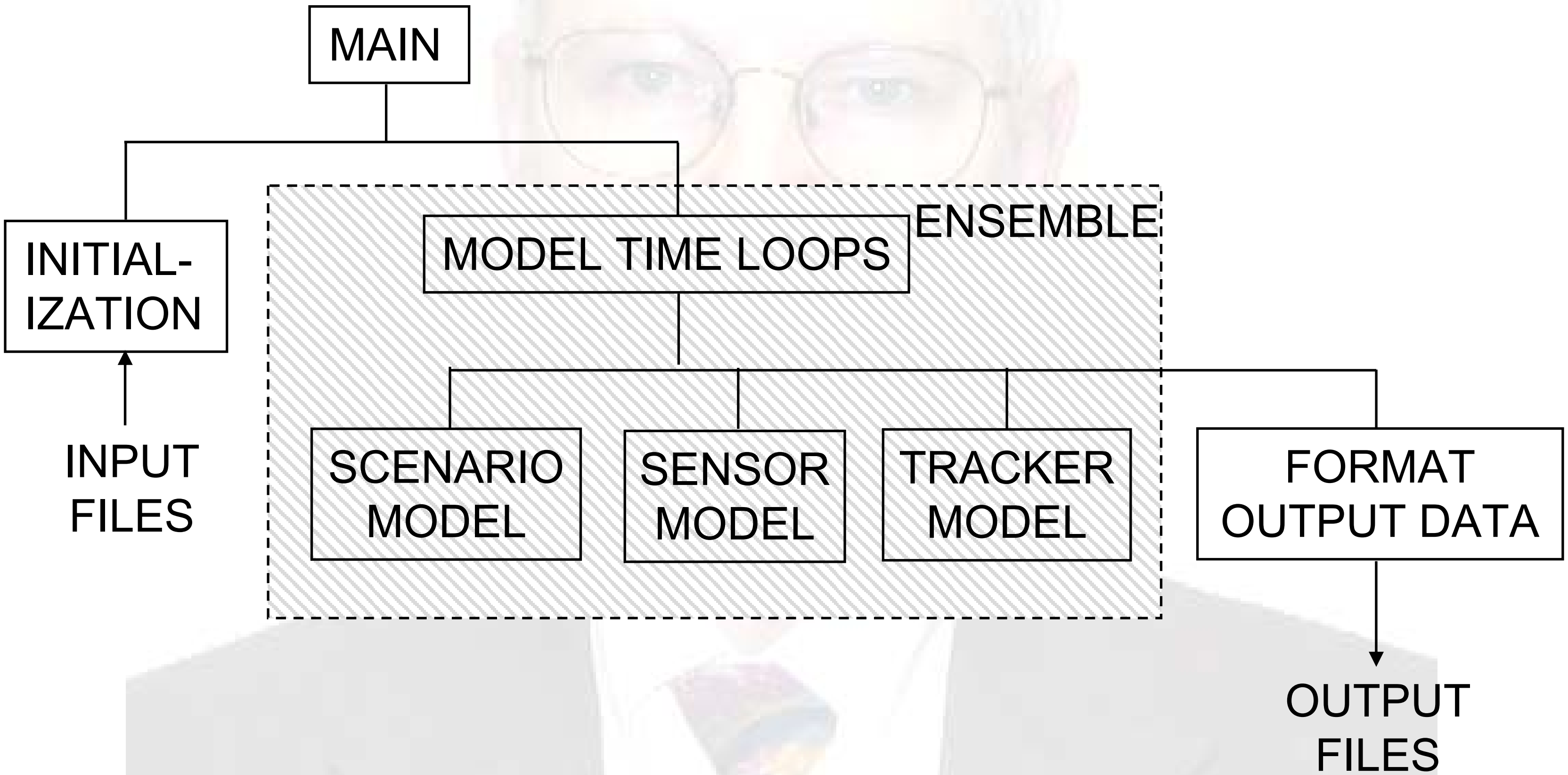
Monte Carlo Simulations Are

- An Organized Collection of Simulations
  - Using Independent Random Numbers
  - Emulative Level or Higher Level
  - Represents Ensemble Statistics
- A Data Collection Function
  - Saves Parameters of Interest from Each Run
  - Detect and Count Anomalies or Special Events
- A Data Reduction and Display Function

Overall Organization

- Top Level
  - Random Number Generator Reseeding
  - Scenario Definition
  - System Configuration Definition
- Ensemble of System Simulations
  - Run in Loops
- Data Collection
  - During and After Each Simulation
- Data Reduction
  - After Runs are Completed

Monte Carlo Module Hierarchy



Each Simulation in Run

- Independent Random Numbers
  - Either Reseed Random Number Generator with Different Seeds
  - Or Do Not Reseed Random Number Generator
- Reseeding Random Number Generator
  - Allows Repeating Run with Different Data Collection and Reduction
  - Rare Anomalies Can Be Repeated

Examples of Data

- Simulation Parameters Collected
  - State Errors
  - Squares and Cross Products of State Errors
  - Upper and Lower Bounds of State Errors
  - Incorrect and Missed Associations
- Data Reduction Numbers
  - State Variances and Correlations
  - Statistics on Tracker Failures
  - Probabilities of Misassociations

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Topic 28: Variance Reduction

Sensor Systems Engineering for the 21st Century

Hypothesis Testing

- Basic Structure
  - Define Hypotheses H_0 and H_1
    - » H_0 is No Mean Present
    - » H_1 is A Mean is Present
  - Define $P(x<X|\ H_0)$
  - Set Threshold T_C

$$P\left(x \le T_C \middle| H_0\right) = C$$

- Test Is

$$x \le T_C \Rightarrow H_0 \text{ to Confidence Level } C$$

Hypothesis Testing Examples

- Detection with Fixed Threshold
  - Hypothesis H_0 is No Target Present
  - Hypothesis H_1 is Target Present
  - Power in FFT Bin Output is Chi-Square 2
  - Confidence Level C is 1- (False Alarm Rate)
- CFAR Threshold
  - Distribution of Cell Average is Chi-Square $2 \cdot M$
  - Threshold Set for Difference of Two Random Variables

Test For Mean Error in a State Variable Estimate

- Student t Test
  - Hypothesis H_0 of No Bias
  - Test Statistic

$$t = \frac{\Delta x(t_i)}{\sqrt{\mathrm{Var}\{\Delta x(t_i)\}}}$$

- Distribution is
  - Student T Distributed if Sample Variance is Used
  - Normally Distributed if True Variance is Used
- Test to Desired Confidence Level

Extreme Value

- For Any Distribution
  - Probability Distribution of Maximum of N Samples

$$P_N\left(x_N \le X\right) = \left(1 - P\left(x \le X\right)\right)^N$$

- Man of Extreme Value is

$$\underline{\bar{x}}_N = \lim_{L \to \infty} L - \int_{-L}^{L} \left(P\left(x \le t\right)\right)^N \cdot dt$$

- Median is More Computable

Structure of Monte Carlo Simulations

- During Each Run, For Each t_i Save

$$\sum \underline{x}(t_i),\ \sum \underline{x}(t_i)\cdot \underline{x}^T(t_i)$$

$$\sum \underline{x}_p \underline{x}_q^T$$

$$\text{Max}\{\underline{x}(t_i)\},\ \text{Min}\{\underline{x}(t_i)\}$$

$$N\{\text{mis-association}\},\ N\{\text{no association}\}$$

- After Runs
  - Compute Means, Variances, Covariances
  - Count Association Anomalies
  - Use Minimum Variance Estimators

Why Variance Reduction

- A Figure of Merit of an Estimate is Mean over Standard Deviation
- Example: CFAR False Alarm Rate
  - MLE Gives Estimate, for K False Alarms in N Trials, as

$$\hat{P}_{FA} = \frac{K}{N}$$

- Estimate has Binary Distribution with Variance

$$\mathrm{Var}\left\{\hat{P}_{FA}\right\} = \frac{P_{FA} \cdot \left(1 - P_{FA}\right)}{N}$$

- Figure of Merit is

$$\frac{\hat{P}_{FA}}{\sqrt{Var\left\{\hat{P}_{FA}\right\}}} = \sqrt{\frac{N \cdot P_{FA}}{1 - P_{FA}}}$$

- Requirement for Accurate Estimate is

$$N >> \frac{1}{P_{FA}}$$

Variance Reduction Techniques

- Definition
  - Structured Variation of Simulation Statistics
    - » Change Results
    - » Obtain Valid Statistics on Low Probability Events
- Techniques
  - Importance Sampling
  - Other – See References (Kleijnen)

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Uses of Variance Reduction

- Designing Association Algorithms
  - Studying Misassociation Statistics
  - Tuning Association Thresholds
- Designing MHTs
  - Evaluating False Alarm Rates
- Determining CFAR Thresholds
- Any Monte Carlo Estimate Involving Rare Events

Basic Principle of Importance Sampling

- Define a New Event and Noise Distribution
  – Mean of New Event is Same as Mean of Old Event
  – Found by Averaging Over New Noise Distribution
  – Variance of New Event Estimate is Much Lower
- Use Monte Carlo Run to Analyze New Event

Definition of Old and New Events and Noise Distributions

- Old: CFAR Threshold Crossing e($\underline{x}$)
  - Function of Set of Noise Values $\underline{x}$
  - Value of 0 for No Crossing, 1 for Crossing

$$P_{FA} = \mathrm{Exp}\{e(\underline{x})\} = \int e(\underline{x}) \cdot p(\underline{x}) \cdot d\,\underline{x}$$

$$\hat{P}_{FA} = \frac{1}{N} \cdot \sum_{i=1}^{N} e(\underline{x}_i)$$

- New: New Noise Values $\underline{y}$, pdf h($\underline{y}$), and Event e*($\underline{y}$)

$$e*(\underline{y}) = \frac{p(\underline{y})}{h(\underline{y})} \cdot e(\underline{y}), \;\; Exp\{e*(\underline{y})\} = P_{FA}$$

$$\hat{P}_{FA} = \frac{1}{N} \cdot \sum_{i=1}^{N} e*(\underline{y}_i)$$

Mathematical Points from 1971 IEE Paper

- Perfect Mapped Noise Distribution

$$h_0(\underline{y}) = \frac{p(\underline{y})}{P_{FA}}, \ \ \underline{y} \in e(\underline{y}) = 1$$

- Requires That
  - P_{FA} Be Known Exactly
  - Subspace of $\underline{y}$ For Which $e(\underline{y})$ Be Known
- Variance of Estimate is Zero
- Hints at Verification Process

Practical New Mapped Noise Distribution

- Select $\underline{y}$
  - Maximize Probability that $e(\underline{y})=1$
  - Make Most Elements of $\underline{y}$ Same as $\underline{x}$
  - Example:
    - » Increase Variance of Rayleigh Noise in CFAR Detection Cell by Factor of E
    - » Noise in cells in CFAR Mask is Unchanged
- Select Magnitude Factor of y_i So That

$$P\left(e\left(\underline{y}\right)\right) \approx 0.5$$

Rationale for Selection of Noise Distribution

- Values of e*(y)

$$e^*\left(\underline{y}\right)\begin{cases} = \dfrac{p\left(\underline{y}\right)}{h\left(\underline{y}\right)}, & e\left(\underline{y}\right) = 1 \\ = 0, & e\left(\underline{y}\right) = 0 \end{cases}$$

- About the Same for Most Threshold Crossings
- e*(y) is Approximately Binomially Distributed
- Variance is Minimized when P(e=1)=0.5

Examples of Probability Density Ratios, One Bin

- Rayleigh Noise (Envelope CFAR)

$$\frac{p(\underline{y})}{h(\underline{y})} = \frac{\frac{F \cdot y_i}{\sigma^2} \cdot \exp\left(-\frac{(F \cdot y_i)^2}{2\sigma^2}\right)}{\frac{y_i}{\sigma^2} \cdot \exp\left(-\frac{y_i^2}{2\sigma^2}\right)}$$

- Exponential Noise (Power CFAR)

$$\frac{p(\underline{y})}{h(\underline{y})} = \frac{\frac{F}{2\sigma^2} \cdot \exp\left(-\frac{F \cdot y_i}{2\sigma^2}\right)}{\frac{1}{2\sigma^2} \cdot \exp\left(-\frac{y_i}{2\sigma^2}\right)}$$

Selection of F

- An Art, Not A Science
- Found by Trial and Error
- Starting Value Estimated From

$$\frac{p(\underline{y})}{h(\underline{y})} = \frac{\frac{F}{2\sigma^2} \cdot \exp\left(-\frac{F \cdot y_i}{2\sigma^2}\right)}{\frac{1}{2\sigma^2} \cdot \exp\left(-\frac{y_i}{2\sigma^2}\right)} \approx 2 \cdot P_{FA}$$

$$F \cdot \exp\left(-\frac{(F-1) \cdot T}{2\sigma^2}\right) \approx 2 \cdot P_{FA},\ T = \text{Threshold}$$

- Vary F to Make P(e($\underline{y}$)=1) About 0.5

Variance Reduction Conclusions

- Importance Sampling
  - Can Be Used to Reduce Time Required in Monte Carlo Runs
  - Reduction by Four Orders of Magnitude or More is Common
- Selection of <u>y</u>
  - Significant Systems Analysis Problem
  - Best Pursued by Reducing Problem to Single Multiplicative Factor on a Few Elements of <u>x</u>

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Radar Trackers and Applications for SAADS

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Topic 29: Interactive Multiple Models

Sensor Systems Engineering for the 21st Century

References

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- Estimation and Tracking: Principles, Techniques and Software, Yaakov Bar-Shalom and Xio-Rong Li, Artech House (1993).

Definition of IMM

- Use of Multiple State Propagation Models
  - Kalman filter is updated using both sets of extrapolated states and covariances
  - Multiple Kalman filters are carried between updates without bifurcating (splitting) the track file
- Use of Data from Measurements to Obtain Either
  - Weighted average of states and covariances
  - Selection between the models

Use of IMM

- Typically Used to Model Different Target Maneuver Behaviors
- Benefit Tracker Accuracy Because
  - Separate, more accurate models are used instead of noisier generic models
  - Each model more accurately models target behavior
- Number of Models
  - Typically range from two to five
  - Three models is most common

IMM Technology

- Basis is a Markov Model of Target Motion Modes
- Each target motion mode is represented by one of the multiple models
- The probabilities corresponding to the Markov modes are characterized as the PM matrix (illustrated for 3 models):

$$PM = \begin{bmatrix} P(S1 \to S1) & P(S1 \to S2) & P(S1 \to S3) \\ P(S2 \to S1) & P(S2 \to S2) & P(S2 \to S3) \\ P(S3 \to S1) & P(S3 \to S2) & P(S3 \to S3) \end{bmatrix}$$

Steps in Implementing IMM (1 of 2)

- Data in Track File
  - State vector and covariance estimates
  - The set $\underline{\mu}(-)$ of target motion mode probabilities
- Extrapolate and Update Kalman Filter for Each of *r* Target Motion Modes
  - Find each state vector and covariance estimate
  - Save the association likelihoods Λ_j

$$\Lambda_j = \frac{1}{(2\pi)\cdot\left|E_j\right|^{1/2}}\cdot\exp\left(-\frac{1}{2}\cdot\underline{e}_j^T\cdot E_j^{-1}\cdot\underline{e}_j\right),\ \ j = 1\ldots r$$

$$\underline{e}_j = \underline{y} - \underline{h}\left(\underline{\tilde{x}}_j\right),\ \ E_j = H\cdot\tilde{P}_j\cdot H^T + R$$

Steps in Implementing IMM (2 of 2)

- Compute Relative Probabilities of Each Target Motion Mode
  – Find *a priori* probabilities $\underline{c}$ of modes at update
    $$\underline{c} = PM \cdot \underline{\mu}(-)$$
  – Find Bayseian probabilities of each mode μ_j using update likelihood for each model
    $$cn = \sum_{j=1}^{r} \Lambda_j \cdot c_j, \quad \mu_j = \frac{1}{cn} \cdot \Lambda_j \cdot c_j$$
- Compute Interactively Weighted Mean State, Covariance

$$\hat{\underline{x}} = \sum_{j=1}^{r} \mu_j \cdot \hat{\underline{x}}_j, \quad \Delta\hat{\underline{x}}_j = \hat{\underline{x}} - \hat{\underline{x}}_j$$

$$P = \sum_{j=1}^{r} \mu_j \cdot \left(P_j + \left(\Delta\hat{\underline{x}}_j\right) \cdot \left(\Delta\hat{\underline{x}}_j\right)^T\right)$$

IMM Design Process

- Select Target Motion Modes
- Assign Markov Probabilities
  - Define constants from target concepts, OR
  - Set up algorithm to define from scenario
- Develop Kalman Extrapolation and Update for Each Mode
  - Number of states may differ
  - Process noise will differ

IMM Example

- Parsed Singer Motion Model, r=4
  - » No maneuver, probability P_1
  - » Hard turn left, acceleration A, probability $P_2/2$
  - » Hard turn right, acceleration A, probability $P_2/2$
  - » Random lateral acceleration, probability P_3
- Define Markov Probability Matrix PM

$$PM = \begin{bmatrix} .7 & .1 & .1 & .1 \\ .3 & .3 & .3 & .1 \\ .3 & .3 & .3 & .1 \\ .4 & .1 & .1 & .4 \end{bmatrix}$$

Process Noise for Each Mode

- Non-Maneuvering: Zero
- Hard Turn Left: Zero
- Hard Turn Right: Zero
- Random Maneuvering: $A^2/6$
- Compares to Single Model: $A^{2}\cdot(P_2+P_3/6)$
- Result: Improved Accuracy

Using IMMs

- Key is Selection of Models and the Markov Matrix
  - Minimize number of models used
  - Vary the mix in use at a given time with the scenario
  - Compute the Markov matrix adaptively
    - » Function of update time
    - » Function of scenario
- IMM Use is an Art, Not a Science

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Topic 30: Implementing Batch Estimators

Sensor Systems Engineering for the 21st Century

Notation of Batch Estimators

- Method
  - Quadratic Cost Function in $\underline{y}$
  - Nonlinear Problem Statements
    - » Recursion is necessary not one pass like a Kalman update
    - » Result is asymptotically unbiased and efficient
    - » Iteration is multivariate Newton's method
- MLE is the Basis for Formulation

Data Model

$$\underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \underline{y}_m \end{bmatrix} = \begin{bmatrix} \underline{h}(t_1, x(t_1)) \\ \underline{h}(t_2, x(t_2)) \\ \vdots \\ \underline{h}(t_m, x(t_m)) \end{bmatrix} + \begin{bmatrix} \underline{v}_1 \\ \underline{v}_2 \\ \vdots \\ \underline{v}_m \end{bmatrix}$$

$$H = \begin{bmatrix} H_1 \\ H_2 \cdot [\Phi(t_2, t_1)]^{-1} \\ \vdots \\ H_m \cdot [\Phi(t_m, t_1)]^{-1} \end{bmatrix}$$

Numerical Formulation

- The Cost Function

$$J = -\frac{M}{2} \cdot \ln(2\pi) - \frac{1}{2} \cdot \ln(|R|) - \frac{1}{2} \cdot \underline{e}^T \cdot R^{-1} \cdot \underline{e}, \quad \underline{e} = \underline{y} - \underline{h}(\underline{\hat{x}})$$

- The Likelihood Equation

$$\underline{l}(\underline{\hat{x}}) = H^T \cdot R^{-1} \cdot \left(\underline{y} - \underline{h}(\underline{\hat{x}})\right) = \underline{0}$$

- Taylor Series

$$\underline{h}(\underline{\hat{x}}) = \underline{h}(\underline{x}) + H \cdot (\underline{\hat{x}} - \underline{x}) + O\left\{\Delta \underline{x}^3\right\}$$

- Other Variable Changes

$$R^{-1/2} \cdot H = A, \quad R^{-1/2} \cdot \left(\underline{y} - \underline{h}(\underline{x})\right) = \underline{e}$$

- Linearized Likelihood Equation

$$A^T \cdot (\underline{\hat{x}} - \underline{x}) = A^T \cdot A \cdot \underline{e}$$

The Nonlinear Recursion

- Basic Relationship for Recursion

$$\underline{\hat{x}}_i = \underline{\hat{x}}_{i-1} - \left[\frac{\partial^2 J\left(\underline{\hat{x}}_{i-1}\right)}{\partial \underline{\hat{x}}_{i-1}^2}\right]^{-1} \cdot \left[\frac{\partial J\left(\underline{\hat{x}}_{i-1}\right)}{\partial \underline{\hat{x}}_{i-1}}\right]$$

- Can Be Difficult
  - Increasingly Sensitive to Initialization as Amount of Data Increases
  - Selection of Coordinate System is Important

Numerical Method

- Nonlinear Iteration

$$A \cdot \left(\underline{\hat{x}}_{i+1} - \underline{\hat{x}}_i \right) = \underline{e}$$

- Triangularize A with Sequence of Householder Transformations T

$$T \cdot A \cdot \left(\underline{\hat{x}}_{i+1} - \underline{x}_i \right) = \begin{bmatrix} S \\ 0 \end{bmatrix} \cdot \left(\underline{\hat{x}}_{i+1} - \underline{x}_i \right) = T \cdot \underline{e} = \begin{bmatrix} \underline{e'} \\ \underline{e''} \end{bmatrix}$$

- Solve Square Recursion

$$\underline{\hat{x}}_{i+1} = \underline{\hat{x}}_i + S^{-1} \cdot \underline{e'}$$

Use of an MLE

- Initialize Using First Few Measurements
- Update Estimate as New Data Becomes Available
- Extrapolate States Between Updates
  - If States Change with Time
  - Use Closed Form Solution to State Equation
- If Necessary to Re-Initialize Later
  - Use Minimum Sparse Data
  - Add Interspersed Data in Stages

Initialization of an MLE

- Initialization
  - Use First Few Measurements
    - » Compute State Estimates Algebraically
    - » Compute Covariance Matrix
      - State Vector Estimate Computation
      - Variance Propagation Equation
  - Initialize Unobservable States
    - » Best Guess
    - » Large Covariance
- Never Use Ad Hoc Initialization

Necessary *Ad* Hoc Technique: Re-Initialization

- Use Minimum Number of Measurements
  - For Observability of Most or All States
  - Use Either
    - » Algebraic Closed Form
    - » Nonlinear Recursion
- Select Data for State Observability
  - First, Quartile Points, Last Data Point
- Add Data Points in Stages
  - 1/8, 3/8, 5/8, 7/8 Stage, Etc.

Necessary *Ad Hoc* Technique: Control of First Convergence

- Put Hard Limits on State Excursion
  - Visualize Locus of Estimate in State Space During Convergence
  - Allow an Order of Magnitude Beyond Reasonable Limits for Locus
  - Preserve the Direction of $\Delta \underline{x}$
- Limit the Number of Iterations
  - Recursion May Not Converge Early in Run
  - Wait for More Data

Generalizations

- A Priori Data
  - Augment Original Data Equation
  - Treat Previous Data as Pseudo-Measurements
- Weighted Least Squares
  - Replace R^{-1} with W
  - Proceed as Above

Colored Measurement Noise

- Illustrated here with covariance of correlated noise only
- Other measurements will usually be interspersed with correlated measurements
- Example: Servo error in target direction cosines
- Suggestion: Augment all measurements into a single Markov I process

Canonic Variates

- Definition
  - Linear combination of states <u>x</u>
  - Unitless, covariance of I
- General Form

$$\underline{z} = S' \cdot \left(\underline{x}\right),\ \ S' = C \cdot S,\ \ C^T \cdot C = I$$

- Estimation in Canonic Coordinates
  - Optimal numerical conditioning
  - Natural consequence of recommended process

Gelb Example 1-1 PP. 5-6

- Two Sensors
  - Each taking a single noisy measurement
  - $z_i = x + v_i$
  - Unknown x is a constant
  - Measurement noises v_i are unbiased, uncorrleated
- Begin with Linear Estimator

$$\hat{x} = k_1 \cdot z_1 + k_2 \cdot z_2$$

Gelb Example 1-1 (Continued)

- Define error $\tilde{x} = \hat{x} - x$
- Make k_1 and k_2 Independent of x
- Make Mean of Estimate x; Result is $k_1 + k_2 = 1$
- Minimum Squared Error; Result is

$$k_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}, \quad k_2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

- Variance σ^2 of Optimal Estimate is

$$\frac{1}{\sigma^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

The Example from Gelb

- An *ad hoc* Estimation Procedure of a Mean
- Premises are
  - Estimator is Independent of Estimate
  - Estimate is Unbiased
  - Estimate has Minimum Variance for Assumed Form
  - Variances of Measurements are Known
- Result is a Simple Weighted Average

Problem 1-1

- Same as Example Except Variables are Correlated
- Still, $k_1 + k_2 = 1$
- But,

$$E\left(\tilde{x}^2\right) = k_1^2 \cdot \sigma_1^2 + 2 \cdot k_1 \cdot k_2 \cdot \sigma_1 \cdot \sigma_2 \cdot \rho + k_2^2 \cdot \sigma_2^2$$

Solving for Gains

- The Gradient with Respect to k_1 is

$$2 \cdot k_1 \cdot \sigma_1^2 + 2 \cdot (k_2 - k_1) \cdot \sigma_1 \cdot \sigma_2 \cdot \rho - 2 \cdot k_2 \cdot \sigma_2^2 = 0$$

- The Solution for the Gain is

$$k_1 = \frac{-\sigma_1 \cdot \sigma_2 \cdot \rho + \sigma_2^2}{\sigma_1^2 - 2 \cdot \sigma_1 \cdot \sigma_2 \cdot \rho + \sigma_2^2}$$

$$k_2 = \frac{\sigma_1^2 - \sigma_1 \cdot \sigma_2 \cdot \rho}{\sigma_1^2 - 2 \cdot \sigma_1 \cdot \sigma_2 \cdot \rho + \sigma_2^2}$$

When ρ is 1

- What is ρ
  - Called the Correlation Coefficient
  - Is the Normalized Covariance of Two Random Variables
- Meaning of $\rho = \pm 1$

$$v_2 = \pm \frac{\sigma_2}{\sigma_1} \cdot v_1$$

Let’s Use Vectors

- Measurements (unknown x is a scalar)

$$\underline{z} = x \cdot \underline{1} + \underline{v},\ \ Cov\{\underline{v}\} \equiv Exp\left\{\underline{v} \cdot \underline{v}^T\right\} = R$$

$\underline{1} \equiv$ [vector of all 1's]

- Estimate

$$\hat{x} = K \cdot \underline{z}$$

- Estimation Error

$$\tilde{x} = K \cdot \underline{z} - x = K \cdot \underline{v} + K \cdot \underline{1} \cdot x - x$$

Estimator Derivation

- Unbiased Condition

$$K \cdot \underline{1} = 1$$

- Variance of Estimate

$$Exp\left\{\tilde{x}^2\right\} = K \cdot R \cdot K^T - \lambda \cdot \left(K \cdot \underline{1} - 1\right)$$

- Gradient of Error Variance With Respect to Weights K

$$2 \cdot R \cdot K^T - \lambda \cdot \underline{1} = \underline{0}$$

The General Solution

- The Weights

$$K^T = \frac{\lambda}{2} \cdot R^{-1} \cdot \underline{1}$$

- The Unbiased Condition

$$K \cdot \underline{1} = \frac{\lambda}{2} \cdot \underline{1}^T \cdot R^{-1} \cdot \underline{1} = 1$$

- The Value of the Extra Parameter λ

$$\lambda = \frac{2}{\underline{1}^T \cdot R^{-1} \cdot \underline{1}}$$

The General Result

- The Weights

$$K^T = \frac{1}{\underline{1}^T \cdot R^{-1} \cdot \underline{1}} \cdot R^{-1} \cdot \underline{1}$$

- The Resulting Variance of the Estimate

$$Exp\left\{\tilde{x}^2\right\} = K \cdot R \cdot K^T = \frac{1}{\underline{1}^T \cdot R^{-1} \cdot \underline{1}}$$

A Well Known Result

- When All the Cross Terms of R are Zero

$$K^T = \frac{\left[\frac{1}{\sigma_i^2}\right]}{\sum_{i=1}^{N} \frac{1}{\sigma_i^2}} \cdot \underline{1}, \quad \left[\frac{1}{\sigma_i^2}\right] = Diag\left\{R^{-1}\right\}$$

$$Exp\left\{\left(\tilde{x} - x\right)^2\right\} = K \cdot R \cdot K^T = \frac{1}{\sum_{i=1}^{N} \frac{1}{\sigma_i^2}}$$

Method of Maximum Likelihood

- Correlated Noisy Measurements of Constant

$$\underline{y} = z \cdot \underline{1} + \underline{v}, \ \ \mathrm{Cov}\left\{\underline{v}\right\} = R$$

- Log Likelihood Function

$$l(x) = -\frac{N}{2} \cdot \ln\left(2\pi\right) - \frac{1}{2} \cdot \ln\left(\left|R\right|\right)$$

$$-\frac{1}{2} \cdot \left(\underline{y} - x \cdot \underline{1}\right)^{T} \cdot R^{-1} \cdot \left(\underline{y} - x \cdot \underline{1}\right)$$

- Likelihood Equation

$$\frac{\partial l(x)}{\partial x} = \underline{1}^{T} \cdot R^{-1} \cdot \left(\underline{y} - \hat{x} \cdot \underline{1}\right) = 0$$

Solution and Covariance

- Solution to Likelihood Equation

$$\hat{x} = \frac{\underline{1}^T \cdot R^{-1} \cdot \underline{y}}{\underline{1}^T \cdot R^{-1} \cdot \underline{1}}$$

- Variance of Estimate is Cramer-Rao Bound

$$\frac{\partial^2 l(x)}{\partial x^2} = -\underline{1}^T \cdot R^{-1} \cdot \underline{1} = \frac{1}{\sigma^2(\hat{x})}$$

$$\sigma^2(\hat{x}) = \frac{1}{\underline{1}^T \cdot R^{-1} \cdot \underline{1}}$$

ATEP SYS12525

Radar Trackers and Applications for SAADS

November 9, 1999

Topic 31: Tensors in Sensor Systems Analysis

Sensor Systems Engineering for the 21st Century

Tensors in Systems Analysis

- Essence of Tensors
  - Indical notation
  - Invariance of quantities in a coordinate transformation
- Examples in Trackers
  - State vector -- contravariant tensor x^i
  - Covariance matrices -- contravariant tensor P^{ij}
  - Gradients -- increment covariant order

Relevance of Tensors

- Vectors and Matrices
  - Simplifications of indical notation
  - Simplifications limit usefulness
    - » Definitions are for one and two subscripts
    - » Extensions are awkward
    - » Example is Talyor series of function of vector
    - » What is the Taylor series of a function of a matrix?
- Tensors are Groups That
  - Are related to invariants in a linear transformation
  - Model physical processes with vectors, matrices, linear and nonlinear transformations, gradients, and integrals

From Raytheon ATEP SYS12525 for SAADS Day 4, November 9, 1999

Kalman Filter Matrices as Tensors

- Kalman Filter Equivalencies

$$\underline{x} \Leftrightarrow x^i, \ \Phi \Leftrightarrow \Phi_i^j$$

$$P \Leftrightarrow P^{ij}, \ P^{-1} \Leftrightarrow Pinv_{ij}$$

$$\underline{y} = \underline{h}(\underline{x}) + \underline{v} \Leftrightarrow y^i$$

$$H = \frac{\partial \underline{h}(\underline{x})}{\partial \underline{x}} \Leftrightarrow H_i^j$$

$$K \Leftrightarrow K_i^j$$

- Transformations Having Tensor Invariance

$$\underline{\tilde{y}} = \underline{h}(\underline{\tilde{x}}), \ \Delta x^i = K_j^i \cdot \left(y^j - h^j\left(\tilde{x}^*\right)\right)$$

$$\underline{z} = P^{-1/2} \cdot \underline{x}, \ z_i = S_{ij} \cdot x^j$$ (Canonic Variates)

Tensors in Estimation Theory

- Probability Distribution Kernels

$$p\left(x^*\right) = \frac{1}{(2\pi)^{M/2}\left|R\right|^{1/2}} \exp\left(-\frac{1}{2}\cdot Rinv_{ij}\cdot x^i \cdot x^j\right)$$

$$\sum_{n=2}^{\infty} \frac{1}{n!}\cdot J_{i1,i2,\cdots in}\prod_{p=1}^{n} x^{ip},\ \ Rinv_{ij} = J_{ij}$$

- Likelihood Equations

$$l_i\left(\hat{x}^*\right) = H_i^j \cdot Rinv_{jk}\cdot\left(y^k - h^k\left(\hat{x}^*\right)\right)$$

$$Pinv_{ij} = H_i^k \cdot Rinv_{kl}\cdot H_j^l$$

Other Applications Benefiting from Tensor Analysis

- INS Systems
  - Gravity modeling
  - Higher order estimation
- Orbital Mechanics
  - Dynamics
  - Trajectory modeling
- Ionospheric Modeling
  - Ray theory
  - Field theory

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Radar Trackers and Applications for SAADS

November 9, 1999

Topic 32: Technology Trends

Sensor Systems Engineering for the 21^{st} Century

Trends in Sensor Technology

- Technology Categories
  - Device
  - Computational
  - Software Engineering Concepts
- Platform
  - Space
  - UAV
- Usage
  - Networked
  - Autonomous

Technology Categories

- Device Technology
  - Transmitter
  - Receiver/processors
- Computational
  - Moore's Law
  - Digital Receivers
- Systems and Software
  - Sensor Fusion
  - Artificial Intelligence
  - Abstraction of Software CSCs

Transmitter Technologies

- Device Technology
  - SiC -- higher efficiencies
  - Integrated T/R modules
  - Low Cost High Performance ESAs
- Receiver/processors
  - Optical processing
  - SiGe, other beyond silicon
  - Integration, Microminiturization and Low Cost
- Space qualified

Computer Technologies

- Moore’s Law
  - Capacity doubles every 18 months
  - Size, weight, power, cost decrease
  - Human neural capacity for $1K by 2020
- Digital receivers
  - Analog to digital conversion at I.F., R.F.
  - All processing past R.F. preselect or first I.F. filter is digital
- Nanoelectronics -- Quantum leap in 2005
- Multiprocessing

Other Device Technologies

- Lasers
  - High speed data links
  - LIDAR
  - Imaging LIDAR
  - Optical processing
- Focal Planes
  - Uncooled mid band EO/IR
  - Larger fields of view, higher resolutions
  - Higher quantum efficiency
- LNAs
  - Enabling technology for active arrays
  - Breakthrough on V_π will allow nanoelectronics

Systems and Software Technologies

- Sensor Fusion
  - Early implementation: track fusion
  - Already a key element of BMC^{3}I
  - Focus of current programs (AMSTE)
  - All sensors: RF, EO/IR, passive
- Artificial Intelligence
  - Early implementation: track association
  - Current state of the art: MHT, IMM
  - Next: adaptive IMM

The Future

- Adaptive Radar
  - Interactive radar modes
    - » Respond to threat, environment, C^2 requirements
    - » Waveforms suit multiple purposes
    - » Trackers -- IMM palette, MHT flow
  - Radar reconfigures
    - » To respond to target type, target density
    - » To respond to weather, EMI or other non-target environment
    - » To meet user demands and requirements
- Trackers
  - Object Oriented Technologies
  - Carnegie-Mellon Institute maturity levels 4+

The Future of System Engineering

- Sensor Alternatives
  - Low cost integrated ESA modules
  - Wideband long word length ADCs at the element level
  - Optoelectronic RF manifolds
  - Digital receivers
- Signal and Data Processing
  - Integrated multiprocessing
  - Inexpensive teraflop capacities

Software Engineering

- Level of abstraction
  - Object oriented -- functions are objects
  - Dynamically reallocated -- adaptive mode configurations
- Level of Sophistication
  - “Operator” will be software
  - Modes and states
    - » Adaptive
    - » Optimized for demands, environment

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Radar Trackers and Applications for SAADS

November 9, 1999

Topic 33: Beard's Law

Sensor Systems Engineering for the 21st Century

Beard's Law

- Given Probability P of
  - Bug in Line of Code
  - Line in Any One Cable
  - Any Other System Element
- Number of Elements N, Then

$$P(\text{bug free}) < \exp(-N \cdot P) \approx 0$$

$$\langle \text{bugs} \rangle = N \cdot P$$

- Your Tools: N and P

Beard's Law in System Engineering

- Minimize Through All Means Available
  - The Length of Programs
  - Probability of a Bug
- Minimizing Program Length
  - Maximize Systems Analysis Effort
  - Leverage Prior Results and the Literature
- Minimizing Bugs
  - Use CMMI Level 4 or Level 5 software engineering
  - Re-Use Code
  - Use Small Modules
  - Use Software Engineering Principles
  - Integrate Software and Systems Engineering

General Principles

- If Your Implementation is Too Complex, You Haven't Done Enough Analysis and Development
- Remember the KISS Principle
- The Ultimate Sophistication is Simplicity

Basic Radar and Tracker Library

- Design and Analysis of Modern Tracking Systems, Samuel Blackman and Robert Popoli, Artech House (1999).
- Multitarget-Multisensor Tracking, Principles and Techniques, Yaakov Bar-Shalom and Xiao-Rong Li, YBS (1995)
- Radar Engineer’s Handbook, Merrill Skolnik, Ed., Mc-Graw-Hill (1993)

Course Material Support Policy

- Course Material Supported
  - For those who completed the course
  - Only through James K Beard
  - Limited to answering questions on material and clarifications
- Not Supported
  - Material outside scope of course as presented
  - New material or topics
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- Procedure
  - Contact James K Beard at jkbeard@ieee.org
  - Availability of free support is necessarily limited
  - Funding may be necessary for significant demands on time
- Want this course updated and tailored to your needs, and taught at your facility? Just ask through jkbeard@ieee.org

The End